NEGATIVE NUMBERS ARE NOT SO SPECIAL!!!

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The standard way of writing numbers in base 10 is to write them with the digits 0 through 9. But is it the only way? The following alternative is fun to think about: imagine we have digits $S, R, Q, P, 0, 1, 2, 3, 4, 5$ instead (so, still ten of them). The digits $0, 1, 2, 3, 4, 5$ represent the same things that we use them for. As for P, Q, R, S, they come before 0, so they represent $-1, -2, -3, -4$ respectively. How do we count using this system?[1]

Well, to add 1 to a number, you increment the last (i.e., least significant, written right-most digit), and if it wraps from 5 to S then you perform the same on the adjacent digit, and so on. So we get $0, 1, 2, 3, 4, 5, 1S, 1R, 1Q, 1P, 10, 11, 12, 13, 14, 15, 2S, 2R$, etc. Notice how "$1S$" denotes what we write 6, but this system and ours come back in sync for 10 to 15, and generally speaking for any number written using only the digits 0 through 5

But a nice thing about the $SRQP012345$ system is that we can count backwards just as easily: to subtract 1 from a number, you decrement the last digit, and if it wraps from S to 5 then you perform the same on the adjacent digit, and so on. So before 0 we get $P, Q, R, S, P5, P4, P3, P2, P1, P0, PP, PQ, PR, PS, Q5, Q4$, etc.: these are the numbers that we write $-1, -2, -3, -4, -5, -6, -7, -8, -9, -10$ and so on ($P0$ is -10).

So the nice feature of this system is that we can write negative numbers using the exact same symbols that we use to write positive ones: there is no need for a special '$-$' sign.

1. Write the following integers, described in the 0123456789 system, in the $SRQP012345$ system:

   (a) 19
   (b) 56
   (c) 1234
   (d) -19
   (e) -56
   (f) -231

2. Write the following integers, described in the $SRQP012345$ system, in the 0123456789 system:

   (a) $12Q$
   (b) RS
   (c) $PS4$
   (d) $1SP$
   (e) $P89$
   (f) $1P2Q$

3. How do you determine if a number written in this system is positive or negative?

4. Can you come up with an algorithm to convert numbers in 0123456789 system to $SRQP012345$ system and back?

[1] This worksheet is based on a twitter thread by Gro-Tsen

5. Fill in the tables addition, subtraction, and multiplication tables below:

$+$	S	R	Q	P	0	1	2	3	4	5
S										
P										
Q										
R										
0										
1										
2										
3										
4										
5										

$-$	S	R	Q	P	0	1	2	3	4	5
S										
P										
Q										
R										
0										
1										
2										
3										
4										
5										

$\times$	S	R	Q	P	0	1	2	3	4	5
S										
P										
Q										
R										
0										
1										
2										
3										
4										
5										

6. Compute the following:

(a) $1Q + R5$

(b) $1Q - R5$

(c) $1Q \times R5$

(d) $1P2Q + 5RQ3$

(e) $1P2Q - 5RQ3$

(f) $1P2Q \times 5RQ3$

7. How do you write $1/3$ and $2/3$ in this system? Which reals have multiple representations (like $0.999999\ldots = 1.000000\ldots$ in our usual system)?

Converting from the 0123456789 system to $SRQP012345$ system

Given a number x expressed in the 0123456789 system, I denote its representation in the $SRQP012345$ system as $(x)_{new}$. Note that $(0)_{new} = 0$, $(1)_{new} = 1$, $(2)_{new} = 2$, $(3)_{new} = 3$, $(4)_{new} = 4$, $(5)_{new} = 5$ $(-1)_{new} = P$, $(-2)_{new} = Q$, $(-3)_{new} = R$, and $(-4)_{new} = S$.

Now, given a number $5 < x < 10$, $(x)_{new} = 1((x-10)_{new})$. For example, if $x = 7$, then $(7)_{new} = 1((7-10)_{new}) = 1((-3)_{new}) = 1R$

Now consider an arbitrary 2-digit number x represented as ab in the 0123456789 system.

Assume $0 < x < 55$. Then the number can be represented as a 2 digit number in the $SRQP012345$ system. If $b \leq 5$, then $(ab)_{new} = ab$. Now if $b > 5$, then notice that $a \leq 4$, and $-4 \leq b - 10 < 0$. Thus, $(ab)_{new} = (a+1)(b-10)_{new}$. For example, if $a = 4$ and $b = 8$, then $(ab)_{new} = (48)_{new} = (4+1)(8-10)_{new} = 5(-2)_{new} = 5Q$.

Given $-44 < x < 0$, we can represent x too in the $SRQP012345$ system. Notice that

$$(-ab)_{new} = \begin{cases} (-a)_{new}(-b)_{new} & \text{if } b \leq 4 \\ (-a-1)_{new}(10-b)_{new} & \text{otherwise} \end{cases}.$$

For example, we can see that $(-37)_{new} = (-3-1)_{new}(10-7)_{new} = (-4)_{new}(3)_{new} = S3$ and $(-34)_{new} = (-3)_{new}(-4)_{new} = RS$.

Now assume ab is between 56 and 99. Then, $ab = 100 + (-x)$ where $-44 < -x < 0$. Thus, $(ab)_{new} = 1(-x)_{new}$. For example, $(66)_{new} = 1(-34)_{new} = 1RS$.

Now consider an arbitrary 3-digit number x represented as abc in the 0123456789 system.

Assume $0 < x < 555$. Then the number can be represented as a 3 digit number in the $SRQP012345$ system. If $bc \leq 55$, then $(abc)_{new} = a(bc)_{new}$. Now if $bc > 55$, then notice that $a \leq 4$, and $-44 \leq b - 100 < 0$. Thus, $(abc)_{new} = (a+1)(bc-100)_{new}$. For example, if $a = 4$ and $bc = 87$, then $(abc)_{new} = (487)_{new} = (4+1)(87-100)_{new} = 5(-13)_{new} = 5PR$.

Given $-444 < x < 0$, we can represent x too in the $SRQP012345$ system. Notice that

$$(-abc)_{new} = \begin{cases} (-a)_{new}(-bc)_{new} & \text{if } bc \leq 44 \\ (-a-1)_{new}(100-bc)_{new} & \text{otherwise} \end{cases}.$$

For example, we can see that $(-375)_{new} = (-3-1)_{new}(100-75)_{new} = (-4)_{new}(25)_{new} = S25$ and $(-338)_{new} = (-3)_{new}(-38)_{new} = RS2$.

Now assume abc is between 556 and 999. Then, $abc = 1000 + (-x)$ where $-444 < -x < 0$. Thus, $(abc)_{new} = 1(-x)_{new}$. For example, $(652)_{new} = 1(-348)_{new} = 1S52$.

I hope, you understand that we can continue doing this and systematically convert bigger and bigger numbers by breaking them down to smaller numbers. Once again, sorry for the confusion towards the end. Also, let me know if I have made any mistakes. Hope you had fun.