Applicability of Complex Analysis

Polya Vector fields

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Talk given at VIT on 28 September 2023

Characters of our story

- Functions $f : \mathbb{C} \to \mathbb{C}$ (or more generally $f : D \to \mathbb{C}$ where D is an open connected subset of $\mathbb{C}$).
- Functions $f : \mathbb{R}^2 \to \mathbb{R}^2$ (or more generally $f : D \to \mathbb{R}^2$ where D is an open connected subset of $\mathbb{R}^2$).
- Two dimensional vector fields on $\mathbb{R}^2$

Differentiablity

We say a function $f : \mathbb{C} \to \mathbb{C}$ is differentiable at a point z_0 if

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Remark: Given a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}^2$, the corresponding function $F : \mathbb{C} \to \mathbb{C}$ is not necessarily differentiable.

Derivative as a linear approximation

- Given vector spaces V and W over a field F, a function $L : V \to W$ is called linear if $L(v_1 + v_2) = L(v_1) + L(v_2)$ and $L(\alpha v_1) = \alpha L(v_1)$ for all $v_1, v_2 \in V$ and $\alpha \in F$.

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- Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map and let $L((1, 0)) = (a, c)$ and $L((0, 1)) = (b, d)$. Then,

$$\begin{aligned} L((x, y)) &= L((x, 0) + (0, y)) = L((x, 0)) + L((0, y)) \\ &= L(x(1, 0)) + L(y(0, 1)) = xL((1, 0)) + yL((0, 1)) \\ &= x(a, c) + y(b, d) = (ax + by, cx + dy) \end{aligned}$$

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- So, if a function $L : \mathbb{R}^2 \to \mathbb{R}^2$ should be linear when viewed as a function from $\mathbb{C}$ to $\mathbb{C}$, $L \circ R_{\frac{\pi}{2}} = R_{\frac{\pi}{2}} \circ L$.

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Thus,

$$\begin{bmatrix} -\frac{\partial v}{\partial x} & -\frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial y} & -\frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} & -\frac{\partial v}{\partial x} \end{bmatrix}$$

That is, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$. Note, that these are the famous Cauchy-Reimann equations.

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- Thus, Tristan Needham calls a complex derivative an amplitwist.

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- If the work done along every closed curve is zero, the field is called a conservative field.
- If V is a conservative field, then there exists some $\phi : D \to \mathbb{R}$ such that $V = \nabla\varphi = \left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}\right)$

Divergence and Curl

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- Divergence theorem states that if D is the region enclosed by γ, then $F(V, gamma) = \int\int_R [\nabla \cdot V] dA$.
- Curl theorem states that if D is the region enclosed by γ, then $W(V, \gamma) = \int\int_R [\nabla \times V] dA$.

Curl-free vector fields

- If V is curl-free on a simply connected domain, then the work done by the vector field along any closed curve is zero.
- Thus, there exists a function $\varphi : D \to \mathbb{R}$ such that

$$V = \nabla\varphi = \left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}\right)$$

.

Curl-free and divergence-free vector fields

- If in addition, the vector field is also divergence-free, then

$$0 = \nabla \cdot \nabla \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}.$$

  Thus, $V = \nabla \varphi$ and φ is harmonic.

- Conversely, if φ is a harmonic function, then $\nabla \varphi$ is curl-free and divergence-free vector field.

Polya vector fields

- Let $f : D \to \mathbb{C}$ be a function with $f(x+iy) = u(x+iy) + iv(x+iy)$ and let γ be a curve contained in D. Then

$$\int_\gamma f(z)dz = (u+iv)(dx+idy) = \int_\gamma udx - vdy + i\int_\gamma vdx + udy$$
$$= \int_\gamma udx + (-v)dy + i\int_\gamma udy - (-v)dx$$
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- By Cauchy's theorem, if D is simply connected and f is analytic, then $\int_\gamma f(z)dz = 0$ for all closed curves γ in D. Thus, the vector field $\overline{f}(x,y) = (u(x,y), -v(x,y))$ has the property $F(\overline{f},\gamma) = 0$ and $W(\overline{f},\gamma) = 0$.

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- Thus, if f is analytic, $\overline{f}$ is curl-free and divergence-free.

Polya vector fields

Conversely, if f is continuously differentiable and and the vector field $\overline{f}$ is curl-free and divergence-free, then

$$\left(0 = \nabla \cdot \overline{f} = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) \implies \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

and

$$\left(0 = \nabla \times \overline{f} = -\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right) \implies \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Thus, f satisfies the Cauchy-Riemann equations and is hence analytic.

Questions?