TALK ON CONTINUITY AT ABACUS, ST. JOSEPH'S UNIVERSITY

DIVAKARAN D, AZIM PREMJI UNIVERSITY

1. Rediscovering continuity

We typically have two intuitive notions of continuity,

1. The graph is unbroken (or connected)
2. Nearby points are mapped to nearby points. Sometimes we qualify this further by saying - you can force the images to be as close as you want (arbitrarily close) by ensuring that the points are sufficiently close.

These notions are imprecise and thus it is hard to verify if ugly functions like the one below are continuous or not - even if we have a gut feeling it is not continuous.

Example 1.1. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be the function* $f(x) = \begin{cases} 0 & \textit{if } x \in \mathbb{Q} \\ 1 & \textit{otherwise} \end{cases}$. *Clearly, it is impossible to draw its graph. As rational numbers are dense, you can find a rational number as close to* π *as you want. But, their images are far apart. So, according to definition 2, the function should be discontinuous.*

One way to make the first intuition precise would be to call all functions that satisfy the intermediate value property (defined below) continuous.

Definition 1 (Intermediate value property)**.** *A function* $f : A \subset \mathbb{R} \to \mathbb{R}$ *is said to satisfy the intermediate value property, if given any points* $a, b \in A$ *such that* $a < b$ *and* $f(a) \neq f(b)$, *for every* y *between* $f(a)$ *and* $f(b)$ *you can find an* $x \in [a, b]$ *such that* $f(x) = y$.

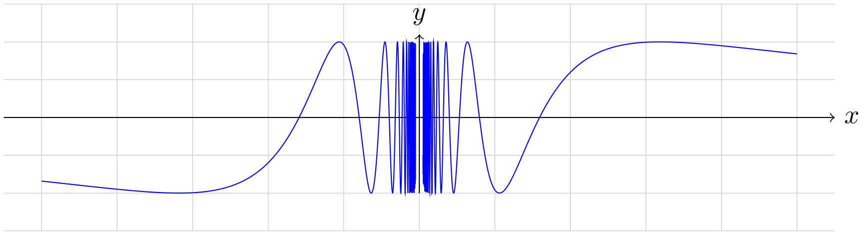
There is a bit of cheating in this translation. We chose to replace the connectedness of the graph, with the connectedness of the image. This is mainly because, connectedness is a much harder concept on the plane than on the line. Notions of connectedness is studies in courses on topology, but is a bit beyond the scope of the present lecture. On the other hand, the second intuition is modified into the standard definition

Definition 2 (Continuity)**.** *A function* $f : A \to \mathbb{R}$ *is said to be continuous at a point* $a \in A$ *if for every* $\varepsilon > 0$, *you can find a* $\delta > 0$ *such that* $|f(x) - f(a)| < \varepsilon$ *if* $|x - a| < \delta$.

There is an even greater cheating in this translation. The intuition of continuity considers the function as a whole, but this definition looks at the behaviour near a point. Mathematicians like to call the first approach global and the second local. This shift from global to local is, in my opinion, is a major shift and is a source of lot of confusion. It is also interesting to note that the generalisation of continuity to a general topological space is defined only globally!

Since the intermediate value property and the standard definition of continuity are rigorous versions of our intuition for continuity, it is natural to ask if the two definitions are equivalent.

Example 1.2 (Discontinuous function that satisfies intermediate value property)**.** *Let* $f : \mathbb{R} \to \mathbb{R}$ *be the function* $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \textit{if } x \neq 0 \\ 0 & \textit{otherwise} \end{cases}$. *Given any value in* $[-1, 1]$, *there are points arbitrarily close to* 0 *whose image takes this value. Thus, the function is not continuous. However, we can show that the function satisfies intermediate value property.*



Example 1.3 (Continuous function that does not satisfy intermediate value property). *Let* $f : [0,1] \cup [2,3] \to \mathbb{R}$ *be the function* $f(x) = x$. *Clearly, the function is continuous. But there is no* $x \in [0,1] \cup [2,3]$ *such that* $f(x) = 1.5$ *even though* $1 = f(1) < 1.5 < f(2) = 2$.

Example 1.4 (Continuous function that does not satisfy intermediate value property). *Let* $f : \mathbb{Q} \to \mathbb{R}$ *be the function* $f(x) = x$. *Once again, you can check this function does not satisfy intermediate value property.*

Question. *Why does the standard definition get priority over intermediate value property?*

The standard definition gets priority because the collection of all functions that satisfy the intermediate value property (IVP) is not well-behaved.

Example 1.5. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be defined as*

$$f(x) = \begin{cases} \frac{1}{x} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

satisfies IVP but is not bounded.

Example 1.6. *(Sum of two functions that satisfies IVP need not satisfy IVP) Let* $f : \mathbb{R} \to \mathbb{R}$ *and* $g : \mathbb{R} \to \mathbb{R}$ *be defined as*

$$f(x) = \begin{cases} \sin^2\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

and

$$g(x) = \begin{cases} \cos^2\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

then,

$$f + g(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Moreover, if the domain is nice (connected), then continuity implies IVP.

Theorem 1 (Intermediate value theorem). *If* $f : [a,b] \to \mathbb{R}$ *is continuous, then given* $a < x_1 < x_2 < b$ *(assume WLOG that* $f(x_1) < f(x_2)$*) and* $f(x_1) < \mu < f(x_2)$. *Then, there exists a point* $x_1 < \nu < x_2$ *such that* $f(\nu) = \mu$.

Proof by Real Induction. Let $S := \{x \in [x_1, x_2] | f(x) < \mu\}$. Then

(1) The set S is non-empty as $x_1 \in S$.
(2) The set S is bounded above as all elements of S are less than or equal to x_2

Thus, S should have a least upper bound, say $c = lub(S)$.

Case 1: $f(c) < \mu$.
Choose $\varepsilon < \mu - f(c)$ and use the definition of continuity to construct a $d > c$ such that $f(d) < \mu$. But, this contradicts the assumption that c is an upper bound. Hence the case is impossible.

Case 2: $f(c) = \mu$.
If $f(c) = \mu$, then we have obtained an x such that $f(x) = \mu$.

Case 3: $f(c) > \mu$.
Choose $\varepsilon < f(c) - \mu$. Then there exists $\delta > 0$ such that $|x - c| < \delta$ implies $|f(x) - f(c)| < \varepsilon$. That is $\mu < f(c) - \varepsilon < f(x) < f(x) + \varepsilon$. Choose $d = c - \frac{\delta}{2} < c$. Then, if $x \in (d, c)$, then $f(x) > \mu$. Also, as c is an upper bound for S, if $x > c$, then $f(x) \geq \mu$. Thus, d is an upper bound for S and $d < c$. Contradicts the assumption that c is the **least** upper bound. Hence this case is impossible. □

2. Sequential criteria for continuity

Once we decide to focus near a point while defining continuity, intuitively continuity means, as you approach the point, the images approach the value at the point. More precisely,

Definition 3 (Sequentially continuous). *A function $f : A \to \mathbb{R}$ is said to be sequentially continuous at a point $a \in A$ if x_n converges to x implies $f(x_n)$ converges to $f(x)$.*

Theorem 2. *Let $f : A \subset \mathbb{R} \to \mathbb{R}$ be a function. Then, f is continuous at a point $a \in A$ iff f is sequentially continuous at a.*

Proof. Assume f is continuous at a. Let x_n be a sequence that converges to a, we will prove $f(x_n)$ converges to $f(a)$. Continuity of f at a implies there exists a $\delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \varepsilon$. Choose this delta. As x_n converges to a, there exists an N such that $|x_n - a| < \delta$ for all $n > N$. Thus, $|f(x_n) - f(a)| < \varepsilon$ for all $n > N$. That is $f(x_n)$ converge to $f(a)$. As, x_n was arbitrary, we have proved that f is sequentially continuous.

Let f be sequentially continuous at a. We will prove f is continuous using a proof by contradition. Assume f is not continuous at a. Then, there exists an $\varepsilon > 0$ such that no matter which δ you choose there exists an x_δ such that $|x_\delta - a| < \delta$ but $|f(x_\delta) - f(a)| \geq \varepsilon$. Choosing $\delta = \frac{1}{n}$, we get a sequence x_n such that $|x_n - a| < \frac{1}{n}$ (that is, x_n converge to a), but $|f(x_n) - f(a)| \geq \varepsilon$ (that is $f(x_n)$ does not converge to $f(a)$). This contradicts the fact that f is sequentially continuous at a. Thus, our assumption that f is not continuous at a has to be wrong. □

3. Back to a global definition

Once we have defined continuity of functions at a point, it is natural to use this definition to define

Definition 4. *A function $f : \mathbb{R} \to \mathbb{R}$ is said to be continuous if it is continuous at all points in A.*

However, this definition still cannot be considered a "global" definition as we are still focusing near points - just that we will do so for all points. The following theorem allows us to give a global definition of continuity.

Theorem 3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then, f is continuous iff inverse image of every open set is a union of open intervals.*

Proof. Assume f is continuous. Let (a, b) be an arbitrary open interval. We will show that $U = f^{-1}((a, b)) = \{x \in \mathbb{R} | f(x) \in (a, b)\}$ is a union of open intervals. Let c be an arbitrary element in U. Then $f(c) \in (a, b)$. Choose $\varepsilon_c = min(b - c, c - a)$. By continuity, there exists a $\delta_c > 0$ such that $|x - c| < \delta_c$ implies $|f(x) - f(c)| < \varepsilon_c$. In other words, if $x \in (c - \delta_c, c + \delta_c)$, then $f(x) \in (f(c) - \varepsilon_c, f(c) + \varepsilon_c) \subset (a, b)$. Thus, $(c - \delta_c, c + \delta_c) \subset U$. Thus, we may write $U = \bigcup\limits_{c \in U} (c - \delta_c, c + \delta_c)$ and hence U is a union of open intervals.

Assume $f^{-1}((a, b))$ is a union of open intervals for all $a, b \in \mathbb{R}$. Then, given a point $c \in \mathbb{R}$, you can write $f^{-1}((f(c) - \varepsilon, f(c) + \varepsilon)) = \bigcup\limits_{\lambda} (a_\lambda, b_\lambda)$. If $c \in (a_{\lambda_0}, b_{\lambda_0})$, then choose $\delta = min(c - a_{\lambda_0}, b_{\lambda_0} - c)$. Clearly, $|x - c| < \delta$ implies $|f(x) - f(c)| < \varepsilon$. As c and ε were arbitrary, f is continuous. □

Due to the importance of sets that can be expressed as union of open intervals, it makes sense to define

Definition 5 (Open sets)**.** *A subset of* $\mathbb{R}$ *is said to be open, if it can be written as a union of open intervals.*

And we already saw that

Theorem 4. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a function. Then the following are equivalent.*

1. *Given any point* $c \in \mathbb{R}$ *and any* $\varepsilon > 0$*, there exists a* $\delta(\varepsilon, c)$ *such that* $|x-c| < \delta$ *implies* $|f(x)-f(c)| < \varepsilon$*.*
2. *Given any point* $c \in \mathbb{R}$ *and any sequence* x_n *converging to* c*,* $f(x_n)$ *converges to* $f(c)$*.*
3. *Inverse image of open sets are open.*

Thus, any of the three statements can be used as the definition of continuity. The three definitions can also be used to generalise the notion of continuity to other domains. The first definition can be generalised to any set where there is a notion of distance - is the content of the subject called metric topology. The second can be generalised to sets on which we have special subsets called open sets (the collection of these subsets should have some good properties) - is the content of the subject called topology. The third definition can be generalised when you can talk about convergence - often we use distance or open sets to define convergence.

4. The singular villain

All my examples were based on $\sin(\frac{1}{x})$ - $\cos(\frac{1}{x})$ has the same qualitative behaviour as $\sin(\frac{1}{x})$. In this section, I will argue, that this is because it is almost impossible to come up with an example that is radically different. However, I cannot make the argument, unless you assume these facts which you will learn in a course on topology.

1. We can define a notion of connectedness on the plane $\mathbb{R}^2$.
2. In addition, there is a notion called local connectedness, which says, if the neighbourhoods of points are connected or not.
3. The graph of $\sin(\frac{1}{x})$ is connected.
4. The graph of $\sin(\frac{1}{x})$ is not locally connected.
5. The graph of $\sin(\frac{1}{x})$ is not closed.

We would ideally like to define continuity as ”the graph is connected”, but $\sin(\frac{1}{x})$ was an obstacle to this definition - we dont want it to be continuous, but its graph is connected. But, if it is even a little bit nicer, then it would become continuous according to our standard definition. More precisely,

Theorem 5. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a function. The following are equivalent.*

1. f *is continuous.*
2. *The graph of* f *is connected and locally connected.*
3. *The graoh of* f *is connected and closed.*

The interested reader should consult the PhD thesis of Michael Ryszard Wojcik titled “Closed and connected graphs of functions; examples of connected punctiform spaces”.