# Is $\frac{dy}{dx}$ a ratio?

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- Warns us about a possible misconception. If $\frac{dy}{dx}$ can be thought of as a fraction because of the notation, why can't we think of $dx$ as a product?
- Also points out that even $\frac{a}{b}$ is not always a fraction - this would come in handy later.

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- In linear algebra, it is cited as an example of linear map.
- But, in a course on differential equations, I had to discuss solving differential equations by separating variables, exact differential equations etc.

## Works like a fraction when you multiply, but not when you add

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- If $y$ is differentiable, then $\frac{dy}{dx}$ behaves like a fraction at least with respect to multiplication and division. The caveat "if $y$ is differentiable" is not so different from the caveat "if $b \neq 0$" in $\frac{a}{b}$.
- But, if we add, we get bizzare stuff!

$$\frac{dy}{dx} + \frac{du}{dv} = \frac{dy.dv + dx.du}{dx.dv}.$$

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- Thus, $\frac{d^2y}{dt^2} = \frac{d^2y}{dx^2}\frac{dx^2}{dt^2} = 6x.4t^2 = 24t^3$.
- On the other hand, as $y = x^3 = (t^2)^3 = t^6$. Therefore $\frac{d^2y}{dx^2} = 30t^4$.

# $\frac{\partial f}{\partial y}$ is not a ratio!

The notation for partial derivatives is quite problematic. The $\partial f$ in $\frac{\partial f}{\partial x}$ is very different from the $\partial f$ in $\frac{\partial f}{\partial x}$. So, you cannot separate the two.

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Let $F(x, y) = 0$. Then implicit differentiation tells us that

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{\partial y}{\partial x}$$

But, as $y$ does not depent on anything other than $x$, the only sensible way to interpret $\frac{\partial y}{\partial x}$ would be $\frac{dy}{dx}$.

# My suggestion to students

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- Though hard to make the notion precise, an expression that can be integrated has some independent existence. They had seen $\int 3x^2 dx$ in Analysis I and $\oint ydx + xdy$ in Analysis III.
- As we do not understand what we are doing, think of it as **jugaad** . Once you get the answer, substitute the value and check if it is indeed a solution.

# Questions?