Complex Analysis

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Complex plane

1.1. Introduction

Most of you are already familiar with complex numbers defined as numbers of the form $a + ib$, where a and b are real numbers. Sums like this are called formal sums - we do not know how to add a and ib so we just call the symbol $a + ib$ the sum of these two numbers. The same idea is used while defining polynomials. The comparison with polynomials also suggests how we can add or multiply two complex numbers.

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$

and

$$(a + ib)(c + id) = ac + i(ad + bc) + i^2bd = (ac - bd) + i(ad + bc).$$

Addition and multiplication so defined satisfy some nice properties making it a field.

Definition 1.1 (Field). A triple $(F, +, .)$ where $+ : F \times F \to F$ and $. : F \times F \to F$ is called a field if + and . satisfy the following conditions

1. Addition is associative: for all $f_1, f_2, f_3 \in F$, $f_1 + (f_2 + f_3) = (f_1 + f_2) + f_3$.
2. Addition is commutative: for all $f_1, f_2 \in F$, $f_1 + f_2 = f_2 + f_1$
3. Existence of additive identity: there exists an element, denoted by 0, such that for all $f \in F$, $f + 0 = f = 0 + f$.
4. Existence of additive inverse: for every $f \in F$, there exists an element $-f$ such that $f + (-f) = 0 = (-f) + f$.
5. Multiplication is associative: for all $f_1, f_2, f_3 \in F$, $f_1.(f_2.f_3) = (f_1.f_2).f_3$.
6. Multiplication is commutative: for all $f_1, f_2 \in F$, $f_1.f_2 = f_2.f_1$.
7. Existence of multiplicative identity: there exists an element denoted by 1 such that for every $f \in F$, $f.1 = f = 1.f$.
8. Existence of multiplicative inverse: For every $f \in F$, there exists an element $\frac{1}{f}$ such that $f. \left(\frac{1}{f}\right) = 1 = \left(\frac{1}{f}\right).f$
9. Multiplication distributes over addition: for all $f_1, f_2, f_3 \in F$, $f_1(f_2 + f_3) = f_1f_2 + f_1f_3$.

Recall that addition and multiplication of real numbers also satisfied these nice properties. The same is also true for the set of complex numbers $\mathbb{C} = \{a + ib | a, b \in \mathbb{R}\}$ under the addition and multiplication defined above. The proof is generally a mixture of the analogous result for real numbers combined with some computation. In what follows, let $z_k = (a_k + ib_k)$. Then, (justify each step)

Lemma 1.2. *Complex addition is associative.*

Proof.

$$\begin{aligned}(a_1 + ib_1) + ((a_2 + ib_2) + (a_3 + ib_3)) &= (a_1 + ib_1) + ((a_2 + a_3) + i(b_2 + b_3)) \\ &= (a_1 + (a_2 + a_3)) + i(b_1 + (b_2 + b_3)) \\ &= ((a_1 + a_2) + a_3) + i((b_1 + b_2) + b_3) \\ &= ((a_1 + a_2) + i(b_1 + b_2)) + (a_3 + ib_3) \\ &= ((a_1 + ib_1) + (a_2 + ib_2)) + (a_3 + ib_3)\end{aligned}$$

☐

Lemma 1.3. *Complex addition is commutative*

Proof.

$$\begin{aligned}(a_1 + ib_1) + (a_2 + ib_2) &= (a_1 + a_2) + i(b_1 + b_2) \\ &= (a_2 + a_1) + i(b_2 + b_1) \\ &= (a_2 + ib_2) + (a + ib)\end{aligned}$$

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Lemma 1.4. *The element* $0 + i0$ *is the unique additive identity*

Proof. First of all notice that $(a+ib)+(0+i0) = (a+0)+i(b+0) = a+ib = (0+a)+i(0+b) = (0 + i0) + (a + ib)$. Thus, $0 + i0$ is an additive identity. Moreover, if $c + id$ is an additive identity, then $(a + ib) + (c + id) = (a + c) + i(b + d) = a + ib$. Thus, $a + c = a$ and $b + d = b$ which implies that $c = 0$ and $d = 0$. ☐

Lemma 1.5. *Given any element* $a + ib$*, the element* $(-a) + i(-b)$ *is its unique additive inverse.*

Proof. To begin with note that $(a + ib) + ((-a) + i(-b)) = (a + (-a)) + i(b + (-b)) = 0 + i0$. Moreover, if $(a + ib) + (c + id) = (a + c) + i(b + d) = 0 + i0$, then $a + c = 0$ and $b + d = 0$. Thus, $c = -a$ and $d = -b$, giving us uniqueness. ☐

Lemma 1.6. *Multiplication is associative*

Proof.

$$\begin{aligned}(a_1 + ib_1)((a_2 + ib_2)(a_3 + ib_3)) &= (a_1 + ib_1)((a_2a_3 - b_2b_3) + i(a_2b_3 + b_2a_3)) \\ &= (a_1(a_2a_3 - b_2b_3) - b_1(a_2b_3 + b_2a_3)) + i(a_1(a_2b_3 + b_2a_3) + b_1(a_2a_3 - b_2b_3)) \\ &= (a_1a_2a_3 - a_1b_2b_3 - b_1a_2b_3 - b_1b_2a_3) + i(a_1a_2b_3 + a_1b_2a_3 + b_1a_2a_3 - b_1b_2b_3) \\ &= ((a_1a_2 - b_1b_2)a_3 - (a_1b_2 + b_1a_2)b_3) + i((a_1a_2 - b_1b_2)b_3 + (a_1b_2 + b_1a_2)a_3) \\ &= ((a_1a_2 - b_1b_2) + i(a_1b_2 + b_1a_2))(a_3 + ib_3) \\ &= ((a_1 + ib_1)(a_2 + ib_2))(a_3 + ib_3).\end{aligned}$$

☐

Lemma 1.7. *Multiplication is commutative*

Proof.

$$(a+ib)(c+id) = (ac-bd)+i(ad+bc) = (ca-db)+i(da+cb) = (c+id)(a+ib)$$

□

Lemma 1.8. *The element* $1+i0$ *is the unique multiplicative identity*

Proof. Firstly, $(a+ib)(1+i0) = (a.1-b.0)+i(a.0+b.1) = a+ib = (1.a-0.b)+i(0.a+1.b) = (1+0i)(a+ib)$. Thus, $1+i0$ is a multiplicative identity. Moreover, if for every $a+ib \in \mathbb{C}$ $(a+ib)(c+id) = (ac-bd)+i(ad+bc) = a+ib$, then $ac-bd = a$ and $ad+bc = b$. Choosing $a = 1$ and $b = 0$, we get $1.c-0.d = 1$, that is $c = 1$. Similarly choosing $a = 0$ and $b = 1$, we get $0.c-1.d = 0$, that is $d = 0$. □

Lemma 1.9. *Given an element* $a+ib \neq 0+i0$*, the element* $\left(\frac{a}{a^2+b^2}\right)+i\left(\frac{-b}{a^2+b^2}\right)$ *is its unique inverse.*

Proof. Note that $(a+ib)\left(\left(\frac{a}{a^2+b^2}\right)+i\left(\frac{-b}{a^2+b^2}\right)\right) = \left(\frac{a^2}{a^2+b^2}-\frac{-b^2}{a^2+b^2}\right)+i\left(\frac{-ab}{a^2+b^2}+\frac{ab}{a^2+b^2}\right) = 1+i0$. Thus, $\left(\frac{a}{a^2+b^2}\right)+i\left(\frac{-b}{a^2+b^2}\right)$ is an inverse. Moreover, if $1+i0 = (a+ib)(c+id) = (ac-bd)+i(ad+bc)$, then $ac-bd = 1$ and $ad+bc = 0$. Multiplying the second equation by b, we get $0 = abd+b^2c$. But, from the first equation, $bd = ac-1$. Thus, $a(ac-1)+b^2c = 0$. That is, $a^2c+b^2c = a$. As $a+ib \neq 0+i0$, $a^2+b^2 \neq 0$. Thus, $c = \frac{a}{a^2+b^2}$. Substituting the value of c in the equation $ad+bc = 0$, we get $ad+b\frac{a}{a^2+b^2} = 0$, which can be simplified to obtain $d = \frac{-b}{a^2+b^2}$. □

Lemma 1.10. *Complex multiplication distributes over addition*

Proof.

$$\begin{aligned}(a_1+ib_1)((a_2+ib_2)+(a_3+ib_3)) &= (a_1+ib_1)((a_2+a_3)+i(b_2+b_3))\\ &= (a_1(a_2+a_3)-b_1(b_2+b_3))+i(a_1(b_2+b_3)+b_1(a_2+a_3))\\ &= (a_1a_2+a_1a_3-b_1b_2-b_1b_3)+i(a_1b_2+a_1b_3+b_1a_2+b_1a_3)\\ &= ((a_1a_2-b_1b_2)+i(a_1b_2+b_1a_2))+((a_1a_3-b_1b_3)+i(a_1b_3+b_1a_3))\\ &= ((a_1+ib_1)(a_2+ib_2))+((a_1+ib_1)(a_3+ib_3)).\end{aligned}$$

□

Putting together all the above lemmas, we have

Theorem 1.11. *The triple* $(\mathbb{C},+,.)$ *is a field.*

The relation with complex numbers and polynomials is actually a lot deeper - it is a good idea to construct complex numbers as $\mathbb{R}[x]/\langle x^2+1\rangle$. The two definition end up being equivalent. Although important, understanding this equivalence is not indispensable, so you may skip it in your first reading.

1.2. Complex numbers as a field of fractions

Let $\mathbb{R}[x]$ be the set of all polynomials with real coefficients. Then one can check (as above) that addition and multiplication of polynomials satisfy all properties except the existence of multiplicative inverse. Such a triple is called a commutative ring. The adjective commutative specifies that multiplication is also commutative. Thus, $(\mathbb{R}[x]+,.)$ forms a commutative ring usually called the ring of polynomials. Moreover, This ring behaves pretty much similarly to the ring of integers.

More precisely, there are special polynomials called irreducible polynomials and every polynomial can be represented uniquely as a product of these irreducible polynomials - up to reordering and multiplication by real numbers. A polynomial is said to be irreducible if it cannot be written as the product of two lower-degree polynomials.

Example 1.12. The polynomial $x^2 + 3x + 2$ is reducible because $x^2 + 3x + 2 = (x + 2)(x + 1)$.

Example 1.13. The polynomial $x^2 + 1$ is irreducible as it cannot be written as the product of two lower-degree polynomials. If we need to write $x^2 + 1$ as a product of two polynomials, both polynomials have to be linear. But if $x^2 + 1 = (ax + b)(cx + d)$ (where $a \neq 0 \neq d$), then $x = -b/a$ or $x = -d/c$ us a root of the polynomial $x^2 + 1$, but we know the polynomial has no real root.

1.2.1. $\mathbb{R}[x]/\langle x^2+1\rangle$ **is a commutative ring.** Given a commutative ring $(R, +, .)$ and a subset I, we are interested in constructing the quotient R/I. The elements of R/I are of course the cosets $r + I = \{r + i | i \in I\}$. Thus, $r + I = s + I$ iff $r - s \in I$. Given, this quotient, it is natural to define the operations + and . on R/I as:

$$(r + I) + (s + I) = (r + s) + I$$

and

$$(r + I).(s + I) = r.s + I.$$

But, there is no guarantee this definition is well-defined - that is the answer might depend on the representative you choose. However, the definition will be well-defined (you can remove this dependence) if I satisfies two conditions:

1. $(I, +)$ forms a subgroup of $(R, +)$.
2. $\forall r \in R$, $\forall i \in I$, $r.i \in I$.

Subsets that satisfy the above conditions are called ideals. Let us verify that if I satisfy the above conditions, then the definition of + and . for R/I are indeed well defined. Let $r + I = r' + I$ and $s + I = s' + I$, is $(r + I) + (s + I) = (r' + I) + (s' + I)$, that is $r - r' \in I$ and $s - s' \in I$. To verify the definitions are well-defined, we need to check $(r + I) + (s + I) = (r' + I) + (s' + I)$ and $(r + I).(s + I) = (r' + I).(s' + I)$. In other words, we need to show $(r + s) - (r' + s') \in I$ and $r.s - r'.s' \in I$. Notice that $(r + s) - (r' + s') = (r - r') + (s - s')$ and both $r - r'$ and $s - s'$ are elements of I. As $(I, +)$ is a group, it is closed under addition and hence, $(r + s) - (r' + s') \in I$. On the other hand, $r.s - r'.s' = r.s - r'.s + r'.s + r'.s' = s(r - r') + r(s - s')$. Now, $s \in R$ and $r - r' \in I$ so $s(r - r') \in I$. Similarly, $r \in R$ and $s - s' \in I$ so $r(s - s') \in I$. And finally by closure under addition, this implies $r.s - r'.s' \in I$.

Exercise 1.14. If R is a commutative ring, show that R/I is also a commutative ring.

Exercise 1.15. Show that $\mathbb{R}[x]$ is a commutative ring.

Exercise 1.16. Show that $\langle x^2 + 1\rangle = \{p(x)|x^2 + 1 \text{ divides } p(x)\}$ is an ideal.

Thus, by the previous exercises, $\mathbb{R}[x]/\langle x^2 + 1\rangle$ is a commutative ring.

1.2.2. $\mathbb{R}[x]/\langle x^2+1\rangle$ **is an integral domain.** Let $p(x)+\langle x^2+1\rangle$ and $q(x)+\langle x^2+1\rangle$ be two elements of the commutative ring $\mathbb{R}[x]/\langle x^2 + 1\rangle$. Then $(p(x) + \langle x^2 + 1\rangle)(q(x) + \langle x^2 + 1\rangle) = 0 + \langle x^2 + 1\rangle$ iff $p(x)q(x) \in \langle x^2 + 1\rangle$. And, $p(x)q(x) \in \langle x^2 + 1\rangle$ iff $x^2 + 1$ divides $p(x)q(x)$.

Theorem 1.17. *$x^2 + 1$ divides $p(x)q(x)$ iff $x^2 + 1$ divides $p(x)$ or $q(x)$.*

Proof. Assume x^2+1 does not divide $p(x)$. Then, using Euclidean algorithm, you can find two polynomials $r(x)$ and $s(x)$ such that $r(x)(x^2+1)+s(x)p(x)=1$. Thus, by multiplying with $q(x)$ on both sides, $r(x)(x^2+1)q(x)+s(x)p(x)q(x)=q(x)$. x^2+1 divides $p(x)q(x)$ so divides both the terms on the left. Hence x^2+1 divides $q(x)$. $\square$

1.2.3. $\mathbb{R}[x]/\langle x^2+1\rangle$ is a 2-dimensional vector space over $\mathbb{R}$.

EXERCISE 1.18. Show that $(\mathbb{R}[x]/\langle x^2+1\rangle,+)$ along with the scalar multiplication $\forall\alpha\in\mathbb{R}$, $\alpha(s+I)=\alpha.s+I$ forms a vector space over $\mathbb{R}$.

Notice that $x^2+I=-1+I$ as $x^2+1\in I$. Similarly, $x^3+I=-x+I$ as $x^3+x=x(x^2+1)\in I$. And, $x^4+I=1+I$ as $x^4-1=(x^2-1)(x^2+1)\in I$.

EXERCISE 1.19. Show that

(1) $x^n+I=1+I$ if $n=4k$ for some natural number k
(2) $x^n+I=x+I$ if $n=4k+1$ for some natural number k
(3) $x^n+I=-1+I$ if $n=4k+2$ for some natural number k
(4) $x^n+I=-x+I$ if $n=4k+1$ for some natural number k

EXERCISE 1.20. Show that 1 and x form a basis for the vector space $\mathbb{R}[x]/\langle x^2+1\rangle$.

1.2.4. $\mathbb{R}[x]/\langle x^2+1\rangle$ is a field. As multiplication is distributive, given $z\in\mathbb{R}[x]/\langle x^2+1\rangle$ the map $L_z:\mathbb{R}[x]/\langle x^2+1\rangle\to\mathbb{R}[x]/\langle x^2+1\rangle$ defined as $L_z(w)=zw$ is a linear map. And Theorem 1.17 tells us that this map is injective. Rank-Nullity theorem tells us that therefore the map should be surjective. That is, there exists some w such that $L_z(w)=1$ or $zw=1$. Hence, z has a multiplicative inverse.

EXERCISE 1.21. Show that the map $\varphi:\mathbb{R}[x]/\langle x^2+1\rangle\to\mathbb{C}$ defined as $\varphi(a+xb)=a+ib$ is a field-isomorphism. More precisely, show that φ is a bijection and $\varphi((a+xb)(c+xd))=\varphi(a+xb)\varphi(c+xd)$

1.3. Complex plane as a vector space

Every field is a vector space over itself. Thus, $\mathbb{C}$ is certainly a vector space over itself. Thus, the more interesting observation is that it forms a vector space over $\mathbb{R}$. Given any real number α, you can define the scalar multiplication of a complex number $a+ib$ by the scalar α as $(\alpha a)+i(\alpha b)$. $(\mathbb{C},+)$ along with the above scalar multiplication turns $\mathbb{C}$ to a vector space over $\mathbb{R}$. It can further be noticed that 1 and i span this vector space. As every element $a+ib$ can be expressed as the linear combination $a.1+bi$. These two vectors are also linearly independent - $a.1+ib=0$ means $a+ib$. Once again, using our analogy with polynomials this is possible iff $a=0=b$.

As any two 2 dimensional vector space are isomorphic, $\mathbb{C}$ is isomorphic to $\mathbb{R}^2$ as a vector space. The linear map $L:\mathbb{C}\to\mathbb{R}^2$ defined as $L(a+ib)=(a,b)$ is a canonical isomorphism. This isomorphism allows us to define multiplication on $\mathbb{R}^2$. Namely, $(a,b).(c,d)=(ac-bd,ad+bc)$.

EXERCISE 1.22. Show that $\mathbb{R}^2$ with usual vector addition and the above product is a field.

However, this is not the only isomorphism one can construct between $\mathbb{C}$ and $\mathbb{R}^2$ - there are infinitely many isomorphisms. Consider the linear map $M:\mathbb{C}\to\mathbb{R}^2$ defined as $M(a+ib)=(a,b/2)$. Thus, the vector (a,b) stand for the complex number $a+i(2b)$ and the vector (c,d) stand for the vector $c+i(2d)$. The product of these two complex numbers is the complex number $(ac-4bd)+i(2(ad+bc))$. This complex number correspond to the vector $(ac-4bd,ad+bc)$. Thus, we could have defined a product $(a,b)\times(c,d)=(ac-4bd,ad+bc)$. Moreover, $(\mathbb{R}^2,+,\times)$ is a field.

Thus, we should realise that there is more than one way in which we can define multiplication on $\mathbb{R}^2$. However, we need not worry too much - these different definitions are somewhat similar. We will explore this story in more detail through exercises.

Exercise 1.23. Show that $(\mathbb{R}^2, +, \times)$ is a field.

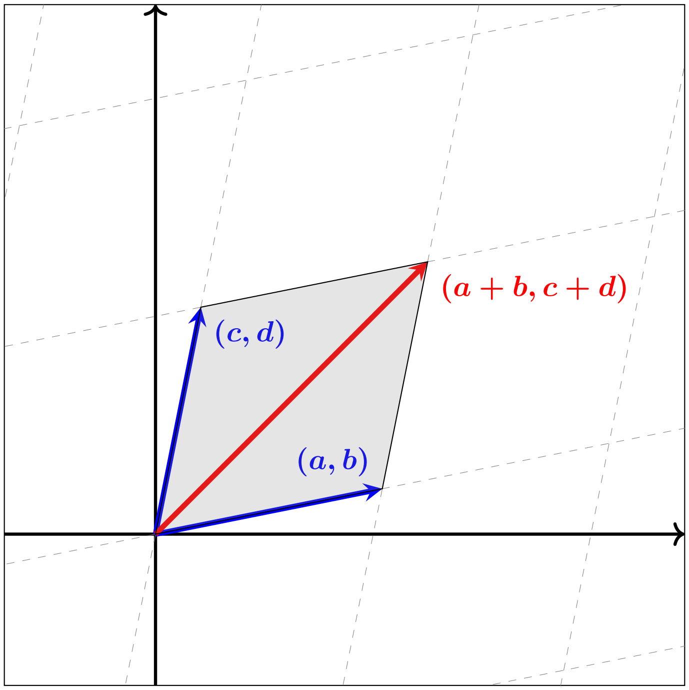
Exercise 1.24. Show that $(\mathbb{R}^2, +, \times)$ is isomorphic to $(\mathbb{R}^2, +, .)$. More precisely, show that there exists an invertible function $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\varphi(v + w) = \varphi(v) + \varphi(w)$ and $\varphi(v \times w) = \varphi(v).\varphi(w)$.

Exercise 1.25. Give an example of another isomorphism from $\mathbb{C}$ to $\mathbb{R}^2$ that gives a third definition of multiplication on $\mathbb{R}^2$

Exercise 1.26. Give an example of an isomorphism from $\mathbb{C}$ to $\mathbb{R}^2$ we have not encountered so far but under this isomosphism you will again obtain the product $(a, b).(c, d)$.

The most natural isomorphism $L : \mathbb{C} \to \mathbb{R}$ defined as $L(a + ib) = (a, b)$ allows us to visualise complex numbers on the plane. This visualisation is the familiar Argand plane description we learn in school. As the elements on the x-axis represent real numbers, they will often be called the real axis. Similarly, as the elements of the y-axis represent purely imaginary numbers, it will also be called the imaginary axis.

Once we have this visualisation, we can make also make sense of the addition law geometrically. Recall, that we defined the addition of vectors to be a coordinate-wise addition. Thus, $(a, b) + (c, d) = (a + c, b + d)$. Notice, that the four points $(0, 0)$, (a, b), (c, d), and $(a + b, c + d)$ form a parallelogram. Vector addition is often defined (or at least motivated) using this parallelogram.



Exercise 1.27. Interpret the subtraction of two complex numbers geometrically.

1.3.1. Inner product. The plane $\mathbb{R}^2$ is a special vector space called an inner product space. An inner product on a vector space V over $\mathbb{R}$ is a map $\langle\ ,\ \rangle : V \times V \to \mathbb{R}$ such that

(1) $\langle x, y\rangle = \langle y, x\rangle$ **(symmetric)**

(2) $\langle x+z, y\rangle = \langle x, y\rangle + \langle z, y\rangle$ **(bi-linear)**

(3) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ iff $x = 0$ **(positive definite)**

EXERCISE 1.28. Show that the usual dot product on $\mathbb{R}^2$ is an inner product.

EXERCISE 1.29. Show that $\langle , \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined as

$$\langle (x_1, y_1), (x_2, y_2)\rangle = \begin{bmatrix} x_1 & y_1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

is an inner product iff

(1) $b = c$, that is the matrix is symmetric.
(2) the determinant $ad - bc > 0$.
(3) the trace $a + d > 0$.

EXERCISE 1.30. Show that if $\langle , \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is an inner product, then there exists a symmetric matrix a $\begin{bmatrix} a & b \\ b & d \end{bmatrix}$ with $ad - b^2 > 0$ and $a + d > 0$ such that

$$\langle (x_1, y_1), (x_2, y_2)\rangle = \begin{bmatrix} x_1 & y_1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

We can use the isomorphism $L : \mathbb{C} \to \mathbb{R}^2$ to transfer the dot product on $\mathbb{R}^2$ to an inner product on $\mathbb{C}$. Namely, we can define $\langle a + ib, c + id\rangle = ac + bd$.

LEMMA 1.31 (Cauchy-Schwarz inequality). *Let V be a vector space and let $\langle , \rangle$ be an innner product on V. Then,*

$$\langle v, w\rangle \leq \|v\|\|w\|.$$

Proof. If v or w is the zero vector, then both sides are the equality would be zero. So, we may assume that $v \neq 0 \neq w$. First consider the case $\|v\| = 1 = \|w\|$. Then

$$\begin{aligned} 0 &\leq \|v - w\|^2 = \langle v - w, v - w\rangle \\ &= \langle v, v - w\rangle - \langle w, v - w\rangle \\ &= \langle v, v\rangle - \langle v, w\rangle - \langle w, v\rangle + \langle w, w\rangle \\ &= \|v\|^2 - 2\langle v, w\rangle + \|w\| \\ &= 2 - 2\langle v, w\rangle \end{aligned}$$

Thus, $\langle v, w\rangle \leq 1 = \|v\|\|w\|$. Now consider the more general case, where $\|v\| > 0$ and $\|w\| > 0$. Then, take $v' = \frac{v}{\|v\|}$ and $w' = \frac{w}{\|w\|}$. Thus, from the earlier case, we have

$$1 \geq \langle v', w'\rangle = \langle \frac{v}{\|v\|}, \frac{w}{\|w\|}\rangle = \frac{1}{\|v\|\|w\|}\langle v, w\rangle.$$

Therefore,

$$\langle v, w\rangle \leq \|v\|\|w\|.$$

□

1.3.2. Norm. As an inner product is positive definite it defines a norm on the vector space.

DEFINITION 1.32. If V is a vector space over $\mathbb{R}$, a norm on V is a function $\| \ \| : V \to \mathbb{R}$ such that

(1) $\|v+w\| \leq \|v\| + \|w\|$ **(Triangle inequality)**
(2) $\|\alpha v\| = |\alpha|\|v\|$ **(Homogeneous)**
(3) $\|v\| = 0$ iff $v = 0$ **(Positive definite)**

EXERCISE 1.33. If $\| \ \|$ is a norm on the vector space V, show that $\|v\| \geq 0$ for all $v \in V$.

LEMMA 1.34. *If $\langle \ , \rangle$ is an inner product on V, then $\|v\| := \langle v, v\rangle^{\frac{1}{2}}$ is a norm.*

Proof. **Triangle inequality:**

$$\begin{aligned} \|v+w\|^2 &= \langle v+w, v+w\rangle \\ &= \|v\|^2 + 2\langle v, w\rangle + \|w\|^2 \\ &\leq \|v\|^2 + 2\|v\|\|w\| + \|w\|^2 \\ &= (\|v\| + \|w\|)^2 \end{aligned}$$

Thus, by taking square root on both sides, we get the triangle inequality.
Homogeneity:

$$\|\alpha v\|^2 = \langle \alpha v, \alpha v\rangle = \alpha^2\langle v, v\rangle = \alpha^2\|v\|^2$$

Thus, by taking square root on both sides, we get the required equality.
Positive definiteness: follows directly from positive definiteness of inner product. □

Question: Given an inner product, there is a canonical norm associated with it. Is every norm induced by an inner product (in this manner)?

This a hard question, but we already saw that we can characterise inner products on $\mathbb{R}^n$. If we knew all the inner products, we may check if this is one among them or not. The basic idea in this characterisation is that **bilinear maps are determined by their action on pairs of basis vectors**. You are familiar with the idea that linear maps are determined by their action on basis vectors and this is analogous to that result.

LEMMA 1.35. *Every linear map $L : \mathbb{R} \to \mathbb{R}$ has the form $L(x) = \alpha x$.*

Proof. Notice that $L(x) = L(x.1) = x.L(1)$. Thus, if we define $\alpha = L(1)$, then $L(x) = \alpha x$ for all x. □

LEMMA 1.36. *Every inner product $\langle\rangle : \mathbb{R}^2 \to \mathbb{R}^2$ has the form $\langle (x_1, y_2), (x_2, y_2)\rangle = \alpha x_1 x_2 + \beta(x_1 y_2 + x_2 y_1) + \gamma y_1 y_2$.*

Proof. Let (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^2$ be fixed but arbitrary. Then,

$$\begin{aligned} \langle (x_1, y_1), (x_2, y_2)\rangle &= \langle (x_1, 0) + (0, y_1), (x_2, y_2)\rangle \\ &= \langle (x_1, 0), (x_2, y_2)\rangle + \langle (0, y_1), (x_2, y_2)\rangle \\ &= \langle (x_1, 0), (x_2, 0) + (0, y_2)\rangle + \langle (0, y_1), (x_2, 0) + (0, y_2)\rangle \\ &= \langle (x_1, 0), (x_2, 0)\rangle + \langle (x_1, 0), (0, y_2)\rangle + \langle (0, y_1), (x_2, 0)\rangle + \langle (0, y_1), (0, y_2)\rangle \\ &= x_1 x_2\langle (1, 0), (1, 0)\rangle + x_1 y_2\langle (1, 0), (0, 1)\rangle + y_1 x_2\langle (0, 1), (1, 0)\rangle + y_1 y_2\langle (0, 1), (0, 1)\rangle \\ &= x_1 x_2\langle (1, 0), (1, 0)\rangle + (x_1 y_2 + x_2 y_1)\langle (1, 0), (0, 1)\rangle + y_1 y_2\langle (0, 1), (0, 1)\rangle. \end{aligned}$$

Define $\alpha = \langle (1,0),(1,0)\rangle$, $\beta = \langle (1,0),(0,1)\rangle$, and $\gamma = \langle (0,1),(0,1)\rangle$ to complete the proof. ☐

Exercise 1.37. Show that $\| \ \|_\infty : \mathbb{R}^2 \to \mathbb{R}$ defined as $\|(x,y)\| = \max(|x|,|y|)$ is a norm.

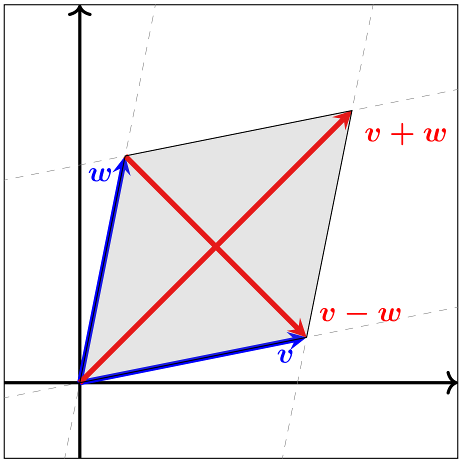
Lemma 1.38. *There does not exist any inner product* $\langle , \rangle$ *on* $\mathbb{R}^2$ *such that* $\|v\|_\infty = \langle v, v\rangle^{\frac{1}{2}}$.

Proof. We will use proof by contradiction. Assume there exists such an inner product $\langle , \rangle$. Then, by Lemma 1.36, we know $\|(x,y)\|_\infty^2 = \langle (x,y),(x,y)\rangle = \alpha x^2 + 2\beta xy + \gamma y^2$ for some α, β, and γ. We will substitute various vectors to discover the values of α, β, and γ. We know $1 = \|(1,0)\|_\infty^2 = \alpha$ and $1 = \|(0,1)\|_\infty^2 = \gamma$. Thus, $\|(x,y)\|_\infty^2 = \langle (x,y),(x,y)\rangle = x^2 + 2\beta xy + y^2$ for some β. From $2 = \|(2,1)\|_\infty^2 = 4 + \beta + 1$, we get $\beta = -3$, but from $3 = \|(3,1)\|_\infty^2 = 9 + \beta + 1$, we get $\beta = -7$. As it is impossible for β to take two different values, our initial assumption should be wrong. ☐

Thus, we have answered our question. More precisely, we showed that there exists norms that are not induced by an inner product. Please note that the proof technique is as important as the result itself. In this proof, we characterised all inner products and showed one specific norm was not equal to the norm induced by any of the inner products. There is another way to go about such proofs. We can show that norms induced by inner products satisfy a property that the given norm does not have. Let us explore this line of thought as well. Given below is a property satisfied by norms induced by an inner product.

Lemma 1.39. *(Parallelogram law) If* $\|v\| = \langle v, v\rangle^{\frac{1}{2}}$ *for all* $v \in V$, *then* $\|v+w\|^2 + \|v-w\|^2 = 2\|v\|^2 + 2\|w\|^2$.

Remark 1.40. The lemma is called called parallelogram because $v+w$ and $v-w$ are the diagonals of the parallelogram that has v and w as adjacent sides. Geometrically, the theorem says the sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of squares of the lengths of its sides.



Proof. Notice that for any v, w

$$\begin{aligned}\|v+w\|^2 &= \langle v+w, v+w\rangle \\ &= \langle v, v+w\rangle + \langle w, v+w\rangle \\ &= \langle v,v\rangle + \langle v,w\rangle + \langle w,v\rangle + \langle w,w\rangle \\ &= \|v\|^2 + 2\langle v,w\rangle + \|w\|^2\end{aligned}$$

Similarly, we can prove

$$\|v-w\|^2 = \|v\|^2 - 2\langle v,w\rangle + \|w\|^2$$

Adding the two, we get the required result $\|v+w\|^2 + \|v-w\|^2 = 2\|v\|^2 + 2\|w\|^2$. ☐

Lemma 1.41. *The norm $\|\|_\infty$ does not satisfy the parallelogram law.*

Proof. Let $v = (1,0)$ and $w = (0,1)$. Then, $\|v\|_\infty^2 = \|(1,0)\|_\infty^2 = 1$, $\|w\|_\infty^2 = \|(0,1)\|_\infty^2 = 1$, $\|v+w\|_\infty^2 = \|(1,1)\|_\infty^2 = 1$, and $\|v-w\|_\infty^2 = \|(1,-1)\|_\infty^2 = 1$. Clearly, $\|v+w\|_\infty^2 + \|v-w\|_\infty^2 = 2 \neq 4 = 2\|v\|_\infty^2 + 2\|v\|_\infty^2$. ☐

In fact, we can construct an infinite family of norms that are not induced by any inner product. But, I leave its proof as an exercise.

Exercise 1.42. Show that $\|(x,y)\|_p = \sqrt[p]{|x|^p + |y|^p}$ is a norm on $\mathbb{R}^2$.

Exercise 1.43. Show that $\| \ \|_p$ satisfies parallelogram law iff $p = 2$.

1.4. Topology of the complex plane

1.4.1. Metric. A metric on a set X is a function $d : X \times X \to \mathbb{R}_{\geq 0}$ such that

(1) $d(x,y) = 0$ iff $x = y$ **(Positivity)**
(2) $d(x,y) = d(y,x)$ **(Symmetry)**
(3) $d(x,y) \leq d(x,z) + d(z,y)$ for all $z \in X$ **(Triangle inequality)**

Lemma 1.44. *Let V be a vector space and $\|\|$ a norm on V. Then, $d(x,y) = \|x-y\|$ is a metric on V.*

Proof. Positive definiteness of norm implies that $0 = \|x-y\| = d(x,y)$ if and only if $x - y = 0$ or $x = y$. Thus, $d(x,y) = 0$ iff $x = y$. As norm is homogeneous $d(y,x) = \|y-x\| = \|(-1)(x-y)\| = |-1|\|x-y\| = \|x-y\| = d(x,y)$. Finally, $d(x,z) = \|x-z\| = \|x-y+y-z\| \leq \|x-y\| + \|y-z\| = d(x,y) + d(y,z)$. ☐

Thus, normed vector spaces form a rich class of examples of metric spaces. In particular, $\mathbb{R}^2$ is a metric space. We also saw there is an infinite family of norms on $\mathbb{R}^2$ and thus on C. These norms give rise to an infinite family of metrics on $\mathbb{C}$ or $\mathbb{R}^2$

Definition 1.45. Define $d_p : \mathbb{C} \to \mathbb{R}$ to be the metric $d_p(z,w) = \|z-w\|_p$

Remark 1.46. These norms and metrics can be generalised to $\mathbb{R}^n$ and thus each $\mathbb{R}^n$ is a metric space. Much of the topology of $\mathbb{R}^2$ or $\mathbb{C}$ that will be discussed in this course can be generalised to $\mathbb{R}^n$.

As every norm induces a metric, it is natural to wonder if every metric is induced by a norm. The following example answers the question.

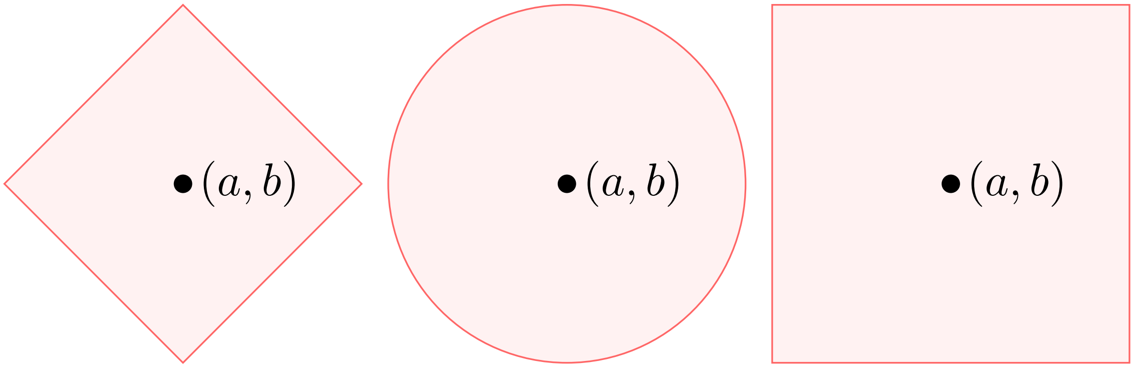
Example 1.47 (Discrete metric). The function $d : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ defined as $d(x,y) = 0$ if $x = y$ and $d(x,y) = 1$ if $x \neq y$ is clearly positive and symmetric. If $x \neq y$, then either $x \neq z$ or $z \neq y$. So, $d(x,z) = 1$ or $d(z,y) = 1$. Thus, $d(x,z) + d(z,y) \geq 1 = d(x,y)$. On the other hand, if $x = y$, then $0 = d(x,y) \leq d(x,z) + d(z,y)$. Hence, d also satisfies the triangle inequality and is thus a metric. If $d(x,y) = \|x-y\|$ for some norm $\|\|$, then $1 = d(2+i0, 0+i0) = \|(2+i0)-(0+i0)\| = \|2+i0\| = \|2(1+i0)\| = 2\|1+i0)\| = 2\|(1+i0)-(0+i0)\| = 2d(1+i0, 0_i0) = 2$, which is a contradiction. Thus, this metric is not induced by a norm.

A metric enables us to define convergence of sequences and and open sets. Thus, we can talk about the continuity of functions $f : \mathbb{C} \to \mathbb{C}$. Such a structure is called a topology.

1.4.2. Open balls and open sets.

Definition 1.48 (Open Ball). Given a metric d on X, we define the open ball centred around x of radius r as $B_d(x,r) = \{y \in X : d(x,y) < r\}$.

Let us understand the definition using some examples. We will primarily look at d_p where $p = 1, 2, \infty$ and the discrete metric on $\mathbb{C}$.



(a) d_1-ball centred at (a,b) **(b)** d_2-ball centred at (a,b) **(c)** d_∞-ball centred at (a,b)

Example 1.49. Let the metric space be $(\mathbb{C}, d_1)$. Then, $d(x+iy, a+ib) = \|(x+iy)-(a+ib)\|_1 = \|(x-a)+i(y-b)\|_1 = |x-a|+|y-b|$. Thus, $B(a+ib,r) = \{x+iy : |x-a|+|y-b| < r\}$. Let us first understand the set $\{x+iy : |x-a|+|y-b| = r\}$. Note that

$$|x-a|+|y-b| = \begin{cases} x-a+y-b & \text{if } x > a \text{ and } y > b \\ a-x+y-b & \text{if } x < a \text{ and } y > b \\ x-a+b-y & \text{if } x > a \text{ and } y < b \\ a-x+b-y & \text{if } x < a \text{ and } y < b \end{cases}.$$

Thus, the set $\{x+iy : |x-a|+|y-b| = r\}$ is the union of four sets

$$\begin{aligned} L_1 &= \{x+iy : x-a+y-b = r, x \geq a, y \geq b\} \\ &= \{x+iy : y = -x+r+a+b, x \geq a, y \geq b\} \\ &= \{x+iy : y = -x+r+a+b, a \leq x \leq r+a\} \end{aligned}$$

$$\begin{aligned} L_2 &= \{x+iy : a-x+y-b = r, x \leq a, y \geq b\} \\ &= \{x+iy : y = x-a+b+r, x \leq a, y \geq b\} \\ &= \{x+iy : y = x-a+b+r, a-r \leq x \leq a\} \end{aligned}$$

$$\begin{aligned} L_3 &= \{x+iy : x-a+b-y = r, x \geq a, y \leq b\} \\ &= \{x+iy : y = x-a+b-r, x \geq a, y \leq b\} \\ &= \{x+iy : y = x-a+b-r, a \leq x \leq a+r\} \end{aligned}$$

$$\begin{aligned} L_4 &= \{x+iy : a-x+b-y = r, x \leq a, y \leq b\} \\ &= \{x+iy : y = -x+a+b-r, x \leq a, y \leq b\} \\ &= \{x+iy : y = -x+a+b-r, a-r \leq x \leq a\} \end{aligned}$$

Thus, the sets L_i are line segments. Moreover, the endpoints of L_1 are $(a, r+b)$ and $(r+a, b)$. The endpoints of L_2 are $(a-r, b)$ and $(a, r+b)$. The endpoints of L_3 are $(a, b-r)$ and $(a+r, b)$.

The endpoints of L_4 are $(a-r,b)$ and $(a,b-r)$. Thus, the union of the four sets is the quadrilateral with vertices $(a,r+b),(a+r,b),(a,b-r),(a-r,b)$. Moreover, it can be seen that the quadrilateral is a rhombus as the length of each L_i is $\sqrt{2}r$. Thus, $B(a+ib,r)=\{x+iy:|x-a|+|y-b|<r\}$ is the interior of this rhombus.

Example 1.50. Let the metric space be $(\mathbb{C},d_2)$. Then, $d(x+iy,a+ib)=\|(x+iy)-(a+ib)\|_2=\|(x-a)+i(y-b)\|_2=\sqrt{(x-a)^2+(y-b)^2}$. Thus, $B(a+ib,r)=\{x+iy:\sqrt{(x-a)^2+(y-b)^2}<r\}=\{x+iy:(x-a)^2+(y-b)^2<r^2\}$.

Example 1.51. Let the metric space be $(\mathbb{C},d_\infty)$. Then, $d(x+iy,a+ib)=\|(x+iy)-(a+ib)\|_1=\|(x-a)+i(y-b)\|_1=\max\{|x-a|,|y-b|\}$. Thus,

$$\begin{aligned}B(a+ib,r)&=\{x+iy:\max\{|x-a|,|y-b|\}<r\}\\&=\{x+iy:|x-a|<r,|y-b|<r\}\\&=\{x+iy:-r<x-a<r,-r<y-b<r\}\\&=\{x+iy:a-r<x<a+r,b-r<y<b+r\}\end{aligned}$$

Example 1.52. Let the metric space be $(\mathbb{C},d_\infty)$ where $d:\mathbb{C}\times\mathbb{C}\to\mathbb{R}$ is defined as $d(x,y)=0$ if $x=y$ and $d(x,y)=1$ if $x\neq y$. Then, if $0<r\leq 1$, then $B(a+ib,r)=\{a+ib\}$ and if $r>1$, then $B(a+ib,r)=\mathbb{C}$.

Definition 1.53. A set $U\subset X$ is said to be open in (x,d) if for every $x\in U$, there exists an $\varepsilon>0$ such that $B(x,\varepsilon)\subset U$.

Example 1.54. Each open ball $B(x,r)\subset X$ is an open set. To see this, notice that given any points $y\in B(x,r)$ let $\varepsilon<r-d(x,y)$. Then if $z\in B(y,\varepsilon)$, then $d(x,z)\leq d(x,y)+d(y,z)\leq d(x,y)+r-d(x,y)=r$

Lemma 1.55. *Arbitrary union of open sets is open. More precisely, if Λ is any set and U_λ is open for all $\lambda\in\Lambda$, then $\cup_{\lambda\in\Lambda}U_\lambda$ is also open.*

Proof. Given any $x\in\cup_{\lambda\in\Lambda}U_\lambda$, $x\in U_{\lambda_0}$ for some $\lambda_0\in\Lambda$. As U_{λ_0} is open, there exists some $\varepsilon>0$ such that $B(x,\varepsilon)\subset U_{\lambda_0}\subset\cup_{\lambda\in\Lambda}U_\lambda$, $x\in U_{\lambda_0}$. As $x\in\cup_{\lambda\in\Lambda}U_\lambda$ was arbitrary, $\cup_{\lambda\in\Lambda}U_\lambda$ is open. ☐

Lemma 1.56. *Finite intersections of open sets is open. More precisely, if $U_1,U_2,\ldots,U_n$ are open sets, then $\cap_{i=1}^n U_i$ is also an open set.*

Proof. Let $x\in\cap_{i=1}^n U_i$. Then $x\in U_i$ for all i. As U_i is open, there exists an ε_i such that $B(x,\varepsilon_i)\subset U_i$. Define $\varepsilon=\min\{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n\}$, then $B(x,\varepsilon)\subset B(x,\varepsilon_i)\subset U_i$. Thus, $B(x,\varepsilon)\subset U_i$ for all i. Thus, $B(x,\varepsilon)\subset\cap_{i=1}^n U_i$. ☐

Example 1.57. Infinite intersections of open sets need not be open. $(-\frac{1}{n},\frac{1}{n})$ is an open subset of $\mathbb{R}$, but $\cap_{i=1}^\infty(-\frac{1}{n},\frac{1}{n})=\{0\}$ and $\{0\}$ is not an open set.

1.4.3. Convergence of sequences.

Definition 1.58. Let (X,d) be a metric space. A sequence $x_n\in X$ is said to converge to a point $x\in X$ if $d(x_n,x)$ converges to 0.

Lemma 1.59. *In the metric space $(\mathbb{C},d_2)$, a sequence $z_n=x_n+iy_n$ converges to a point $z=x+iy$ iff x_n converges to x and y_n converges to y.*

Proof. Suppose z_n converges to z. Then $d_2(z_n, z) = \sqrt{(x_n - x)^2 + (y_n - y)^2}$ converges to zero. But, $|x_n - x| = \sqrt{(x_n - x)^2} \leq \sqrt{(x_n - x)^2 + (y_n - y)^2}$. Thus, $|x_n - x|$ converges to 0. Similarly, we can prove $|y_n - y|$ converges to 0. On the other hand, if x_n converges to x and y_n converges to y, then $|x_n - x|$ and $|y_n - y|$ converges to zero. Therefore, $d_2(z_n, z) = \sqrt{(x_n - x)^2 + (y_n - y)^2}$ converges to zero. ☐

Convergence of sequences and open sets are both tools that allow us to define continuity. Thus, it is only natural to expect that the two are related to each other. This relationship is described by the following lemma.

LEMMA 1.60. *In a metric space (X, d) a sequence x_n converge to a point x iff given any open set U such that $x \in U$, there exists an $\varepsilon > 0$ such that $B(x, \varepsilon)$ contains all but finitely many x_n.*

Proof. Suppose x_n converges to x. Consider any open set U containing x. Then, as U is open, there exists an $\varepsilon > 0$ such that $B(x, \varepsilon) \subset U$. But, $d(x_n, x)$ converges to 0. Therefore, there exists N such that $0 \leq d(x_n, x) < \varepsilon$ for all $n \geq N$. That is, all but finitely many elements x_n belong to $B(x, \varepsilon)$ and thus belong to U.

Suppose for every open set U containing x, there exists an $\varepsilon > 0$ such that $B(x, \varepsilon)$ contains all but finitely many x_n. Given any $\varepsilon > 0$, choose $U = B(x, \varepsilon)$. As $B(x, \varepsilon)$ is an open set containing x, all but finitely many x_n will belong to $B(x, \varepsilon)$. Thus, there exists an N such that $x_n \in B(x, \varepsilon)$ for all $n \geq N$. That is, $d(x_n, x) < \varepsilon$ for all $n \geq N$. Hence, x_n converges to x. ☐

DEFINITION 1.61. In a metric space (X, d) a set K is said to be closed if its complement is open.

THEOREM 1.62. *A subset K of a metric space (X, d) is closed iff given any sequence x_n such that $x_n \in K$ for all n and x_n converges to x, the limit point $x \in K$.*

Proof. Suppose K is closed. Consider any sequence x_n such that $x_n \in K$ and x_n converges to x. We will use a proof by contradiction to prove that $x \in K$. Assume $x \notin K$. The set $U := K^C$ is open and contains x. Thus, by previous result x_n will belong to U for all by finitely many n. But we had taken a sequence x_n such that $x_n \in K$ and thus we have a contradiction. Therefore the assumption that $x \notin K$ has to be wrong. In other words, $x \in K$.

Assume that: given any sequence x_n such that $x_n \in K$ and x_n converges to x, the limit point $x \in K$. We will prove that K is closed by proving that $U := K^C$ is open. Once again, we will use a proof by contradiction. Assume U is not open. Then there exists an element $x \in U$ such that $B(x, r) \not\subset U$ for any r. Therefore $B(x, \frac{1}{n}) \not\subset U$ for any n. That is, for each n, we can find a point x_n such that $x_n \in B(x, \frac{1}{n})$ but $x_n \notin U$ or in other words $x_n \in K$. But as $x_n \in B(x, \frac{1}{n})$, $d(x_n, x) < \frac{1}{n}$. That is x_n converges to x. But, $x_n \in K$ and x_n converges to x therefore $x \in K$. But we assumed this x belongs to U and hence does not belong to K - that is, we have a contradiction. ☐

1.5. More geometry: Complex numbers as funcitons

Earlier, we saw a geometric view of complex addition, where we used the identification of complex numbers with $\mathbb{R}^2$. We could also identify the set of complex numbers with some other sets to get interesting geometric ways of looking at complex numbers. To appreciate this completely, you would need a deeper understanding of Euclidean geometry.

Recall that we say two triangles are called congruent if you can move (using rotations, reflections and translation) one triangle onto the other. Similarly, we say two triangles are similar if you can we can transform one triangle to another using rotations, reflections, translations, **and dilations**. These four types of maps, generate a class of functions called affine maps.

Definition 1.63 (Affine map). A map $f : \mathbb{R}^2 \to \mathbb{R}^2$ of the form $f(a,b) = (ax+by+c, dx+ey+f)$ is called an affine map.

You may recall from school that the study of congruence and similarity formed a major part of the Euclidean geometry you learnt in school. And, as mentioned earlier, two triangles are similar iff there is an affine map that takes one triangle to another. It is a much deeper fact that every aspect of Euclidean geometry is the study of some property that is preserved under affine maps. This observation led Felix Klein to characterise geometry as the study of properties that are preserved under some collection of maps. And this led to other forms of geometry like spherical and hyperbolic geometry.

The aim of the digression was to convey the importance of studying affine maps. We will denote the set of affine maps as $\mathcal{A}$ and we will identify the set of complex numbers with subsets of $\mathcal{A}$. These identifications will provide some deeper insights into complex arithmetic.

1.5.1. Addition as translation. Consider the map $\Phi : \mathbb{C} \to \mathcal{A}$ that takes the complex number $a+ib$ to the map $T_{a+ib} : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $T_{a+ib}(x,y) = (x+a, y+b)$.

Exercise 1.64. Verify that $T_{a+ib} \circ T_{c+ib} = T_{(a+ib)+(c+id)}$[1].

Recall that we had earlier used the identification between $\mathbb{R}^2$ and $\mathbb{C}$ to define a multiplication on $\mathbb{R}^2$. Similarly, we could have used the identification between $\mathbb{C}$ and $\mathcal{A}$ to translate function composition to an operation on $\mathbb{C}$. If we did, we would reinvent complex addition!

Exercise 1.65. Show that Φ is injective.

1.5.2. Multiplication as Amplitwist. Consider the map $\overline{\Psi} : \mathbb{C} \to \mathcal{A}$ that takes the complex number $a+ib$ to the map $L_{a+ib}(x,y) = (ax-by, bx+ay)$. I am not doing anything magical, I am just taking the product $(a,b).(x,y)$ defined earlier. Notice that L_{a+ib} is a linear map. If you prefer thinking of linear maps as matrices, you could also consider the map $\Psi : \mathbb{C} \to GL_2(\mathbb{R})$ defined as

$$\Psi(a+ib) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

Exercise 1.66. Verify that $L_{a+ib} \circ L_{c+id} = L_{(a+ib)(c+id)}$[2].

Exercise 1.67. Show that Ψ is injective.

Further notice that the matrix

$$\Psi(a+ib) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = (a^2+b^2) \begin{bmatrix} \frac{a}{a^2+b^2} & \frac{-b}{a^2+b^2} \\ \frac{b}{a^2+b^2} & \frac{a}{a^2+b^2} \end{bmatrix}.$$

Exercise 1.68. Show that

$$\begin{bmatrix} \frac{a}{a^2+b^2} & \frac{-b}{a^2+b^2} \\ \frac{b}{a^2+b^2} & \frac{a}{a^2+b^2} \end{bmatrix} \begin{bmatrix} \frac{a}{a^2+b^2} & \frac{-b}{a^2+b^2} \\ \frac{b}{a^2+b^2} & \frac{a}{a^2+b^2} \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Definition 1.69. A matrix A is called orthogonal if $AA^T = I$. The collection of all 2×2 orthogonal matrices is denoted as $O(2)$. Notice that as $1 = \det(AA^T) = \det(A)\det(A^T) = \det(A)^2$. Thus, $\det(A) = \pm 1$. The collection of all elements of $O(2)$ with determinant 1 is denoted as $SO(2)$.

Exercise 1.70. Given $a+ib \in \mathbb{C}$ verify that $\Psi(z) \in SO(2)$ iff $a^2+b^2 = 1$.

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[1] In the language of group theory, $\Phi : (\mathbb{C}, +) \to (\mathcal{A}, \circ)$ is a homomorphism. You may ignore this remark if you have not seen group theory earlier

[2] That is, Ψ and $\overline{\Psi}$ are homomorphisms

Exercise 1.71. Show that $A \in SO(2)$ iff there exits some $\theta \in [0, 2\pi]$ such that

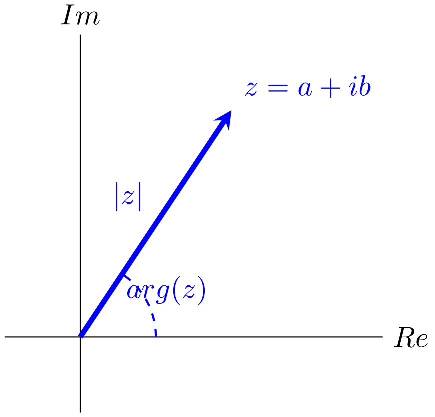
$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

By comparison, we see $\cos(\theta) = \frac{a}{a^2+b^2}$ and $\sin(\theta) = \frac{b}{a^2+b^2}$. That is, $\tan(\theta) = \frac{b}{a}$. Hence θ is the angle (a, b) makes with the x-axis.

Definition 1.72. Given a complex number $a + ib$, define the modulus of $a + ib$, denoted as $|a + ib|$, to the the number $a^2 + b^2$. Given a non-zero complex number $a + ib$, define the argument of $a + ib$, denoted as $arg(a + ib)$, using the formula

$$arg(a+ib) = \begin{cases} \tan^{-1}(\frac{y}{x}) & \text{if } y \geq 0 \text{ and } x \neq 0 \\ \pi + \tan^{-1}(\frac{y}{x}) & \text{if } y < 0 \text{ and } x \neq 0 \\ \frac{\pi}{2} & \text{if } y > 0 \text{ and } x = 0 \\ \frac{-\pi}{2} & \text{if } y < 0 \text{ and } x = 0 \end{cases}.$$

Geometrically, $|a + ib|$ is the length of the vector and $\arg(a + ib)$ is the angle it makes with x-axis.



Let $z = a + ib$, let $r = |a + ib|$, and $\theta = \arg(a + ib)$. Then,

$$\Psi(z) = r\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Thus, $\Psi(z)$ should be thought of as scaling (or amplification) by its modulus and rotating/twisting by its argument. Thus, the operation will be called an amplitwist[3].

Further notice that we can write

$$\begin{aligned} \Psi(z) &= \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \\ &= \Psi(r)\Psi(\cos(\theta) + i\sin(\theta)) \\ &= \Psi(r(\cos(\theta) + i\sin(\theta))). \end{aligned}$$

As Ψ is injective, this implies that $z = r(\cos(\theta) + i\sin(\theta))$.

Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$. Further let $|z_i| = r_i$ and $arg(z_i) = \theta_i$ Then $\Psi(z_1) \circ \Psi(z_2)$ is the linear map that scales by the product r_1r_2 and be rotates by $\theta_1 + \theta_2$. More precisely,

$$\Psi(z_1z_2) = \Psi(z_1)\Psi(z_2) = r_1r_2\begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix}.$$

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[3] A terminology lifted from Visual complex analysis by Tristan Needham

If you know the formula for $\cos(\theta_1+\theta_2)$ and $\sin(\theta_1+\theta_2)$, then you can also verify this algebraically by explicitly verifying

$$\begin{bmatrix}\cos(\theta_1) & -\sin(\theta_1)\\ \sin(\theta_1) & \cos(\theta_1)\end{bmatrix}\begin{bmatrix}\cos(\theta_2) & -\sin(\theta_2)\\ \sin(\theta_2) & \cos(\theta_2)\end{bmatrix} = \begin{bmatrix}\cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2)\\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2)\end{bmatrix}.$$

On the other hand, if the geometric reasoning makes sense to you, the above equation can be used to find the formula for $\cos(\theta_1+\theta_2)$ and $\sin(\theta_1+\theta_2)$. In fact, pretty much all trigonometric formulae can be proved using complex numbers. Check out Page 14 in Visual complex analysis by Tristan Needham for more details.

More importantly, the injectivity of Ψ tells us that $z_1z_2 = r_1r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$. That is, the product of two complex numbers is the complex number whose modulus is the product of the modulus of those two complex numbers and its argument is the sum of the arguments of the two complex numbers. We will see more implications of this observation in the next class.

Recall, that we defined geometry as set of properties that are preserved under some collection of maps. And, we would identify two objects if there is a map in this collection that takes one to another. Now we would want any identification to satisfy the following conditions:

1. Each element should be identified with itself. In other words, the collection should have a map that takes an element to itself. Although, it is possible to attain this in an extremely complicated way, the property would surely be guaranteed if the identity map belongs to the collection.
2. If x is identified with y, then y should be identified with x. In other words, if there is a function f that takes x to y, there is another function that takes y to x. This would be guaranteed if every element of the collection is invertible and if f belongs to the collection so does f^{-1}..
3. If x is identified with y and y is identified with z, then, x is identified with z. This would be guaranteed if given f, g in the collection, $f \circ g$ belongs to the collection. In other words, function composition is a binary operation on this collection of maps.

And these considerations motivate the concept of a group defined below.

Definition 1.73. (Group) A group is a set G equipped with a binary operation, that is, a function $+ : G \times G \to G$ satisfying the following conditions:

1. For all $x, y, z \in G$, $x + (y + z) = (x + y) + z$
2. There exists a special element denoted by 0 such that $x + 0 = x = 0 + x$.
3. For every $x \in G$, there exists a y in G such that $x + y = 0 = y + x$.

In fact, the famous Cayley's theorem tells us that this is a perfect way to think of groups as all groups are isomorphic to some collection of maps. An isomorphism is a map between two groups $(G, +)$ and $(H, .)$ is a bijective function $\varphi : G \to H$ such that $\varphi(g_1 + g_2) = \varphi(g_1).\varphi(g_2)$ (that is the algebraic structure is preserved).

Exercise 1.74. Let $\mathcal{A}$ be the collection of affine maps. Show that $(\mathcal{A}, \circ)$ forms a group.

Exercise 1.75. Show that $(O(2), .)$ and $(SO(2), .)$ (where . represents matrix multiplication) form groups.

Exercise 1.76. Define an operation $\mp : [0, 2\pi) \times [0, 2\pi) \to [0, 2\pi)$ as

$$\mp(a, b) = \begin{cases} a + b \text{ if } a + b < 2\pi \\ a + b - 2\pi \text{ otherwise} \end{cases}$$

Show that $([0, 2\pi), \overline{+}$ is a group.

Exercise 1.77. Show that $(SO(2), .)$ is isomorphic to $([0, 2\pi), \overline{+})$.

Exercise 1.78. Let $\mathbb{R}_{>0} = \{x \in \mathbb{R} | x > 0\}$. Show that $(\mathbb{R}_{>0}, \times)$ forms a group.

Exercise 1.79. Show that $(\mathbb{R}_0 \times [0, 2\pi), *)$ where $*$ is the operation defined as $(r_1, \theta + 1) * (r_2, \theta_2) = (r_1 r_2, \theta_1 + \theta_2)$. This is an example of a direct product of groups and is also represented as $(\mathbb{R}_{>0}, \times) \times ([0, 2\pi), \overline{+})$.

Exercise 1.80. Show that $(\mathbb{C} \setminus \{0\}, \times)$ is a group.

Let us now revisit what we learnt towards the end of last class. We defined a map $\Psi : \mathbb{C} \to \mathcal{A}$ and saw that

$$\begin{aligned}\Psi(z) &= \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \\ &= \Psi(r)\Psi(\cos(\theta) + i\sin(\theta)) \\ &= \Psi(r(\cos(\theta) + i\sin(\theta))).\end{aligned}$$

And as Ψ is injective, this implies that $z = r(\cos(\theta) + i\sin(\theta))$ where $r \in \mathbb{R}_{>0}$ and $\theta \in [0, 2\pi)$.

Exercise 1.81. Show that $F : (\mathbb{C} \setminus \{0\}, \times) \to (\mathbb{R}_{>0}, \times) \times ([0, 2\pi), \overline{+})$ defined as $F(z) = (|z|, arg(z))$ is an isomorphism.

Notice that this matches with the polar coordinate system you might have studied on $\mathbb{R}^2$. Addition had a simple representation in cartesian coordinates and multiplication now has a simple representation in polar coordinates.

The vocabulary we have developed so far lets us summarize what we have studied so far in three lines:

1. $\mathbb{C}$ and $\mathbb{R}^2$ are both vector spaces and $L : \mathbb{C} \to \mathbb{R}^2$ defined as $L(a + ib) = (a, b)$ is an isomorphism. We can use this isomorphism to define a multiplication on $\mathbb{R}^2$.
2. Let $\mathcal{T}$ be the collection of all translations of $\mathbb{R}^2$. And let the map $\Phi : (\mathbb{C}, +) \to (\mathcal{T}, \circ)$ be the map that takes the complex number $a + ib$ to the map $T_{a+ib} : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $T_{a+ib}(x, y) = (x + a, y + b)$. Then, Φ is an isomorphism.
3. $F : (\mathbb{C} \setminus \{0\}, \times) \to (\mathbb{R}_{>0}, \times) \times ([0, 2\pi), \overline{+})$ defined as $F(z) = (|z|, arg(z))$ is an isomorphism.

1.5.3. Conjugate. Let $z = a + ib$. We say a is the real part of z and write $a = Re(z)$. Similarly, we say b is the imaginary part of z and write $b = Im(z)$. Also, note that addition and multiplication corresponded to affine maps. But all these maps had a positive determinant (we call such a map an orientation-preserving map). On the other hand, any reflection has a negative determinant (that is, the map is orientation reversing). So, it is clear that we cannot express reflections using addition and multiplication. You have encountered a unary operation on the complex plane that corresponds to reflection with respect to x-axis, namely, conjugation.

Definition 1.82. Given a complex number $z = a + ib$, the complex conjugate of $a + ib$, denoted as $\overline{a + ib}$, is the complex number $a - ib$.

Clearly, conjugation is related to reflection as the map that sends (a, b) to $(a, -b)$ corresponds to reflection about x-axis. Furthermore, the following exercises explains how to reflect about any other line using reflection about x-axis, rotations, and translations.

Exercise 1.83. Construct the linear map that gives the image under the reflection of a line passing through the origin. Let the line l, pass through the origin and make angle θ to the positive X axis. We break this up into 3 steps and write the matrix corresponding to each linear transformation

1. Step 1. Find the matrix corresponding to the function that maps the line l to X axis. Remember that you have to have to rotate by angle θ clockwise.
2. Step2. Now reflect along X axis. Find the matrix such that a point gets reflected along X axis.
3. Step 3: Now find the matrix that maps the X axis back to the origin line l.
4. Step 4: Compose these matrices to find the matrix corresponding to the function that reflect a point along line l.

Exercise 1.84. Explain how to think of a function that reflects a point about a line l that does not pass through the origin.

So, addition, multiplication and conjugation together capture translation, rotation, dilation and reflection. Thus, all aspects of Euclidean geometry can be captured using these three operations. Moreover, this allows us to view the inner product from the last class in another way! Given two complex numbers $z = a+ib$ and $w = c+id$. Then $\bar{z}w = (ac+bd)+i(ad-bc)$. Notice that $Re(\bar{z}w) = \langle z, w\rangle$. And, if you have studied cross product on $\mathbb{R}^2$, you will notice $|Im(\bar{z}w)| = \|(a, b) \times (c, d)\|$ and the sign captures whether this vector is pointing up or down. Thus, all these ideas are captured by the operations of complex addition, multiplication and conjugation.

Did you know?

Some natural numbers can be expressed as a sum of two squares. Some simple examples are $2 = 1^2 + 1^2$, $25 = 3^2 + 4^2$, and $100 = 8^2 + 6^2$. it is natural to ask the question:

Question: Given two natural numbers, both expressible as a sum of two squares, is the product also expressible as a sum of two squares? In other words, is the collection of all natural numbers that can be expressed as a sum of two squares closed under multiplication?

If one is not lazy, it is quite elementary to solve this question. Notice that if $n = a^2+b^2$ and $m = c^2+d^2$, then it is quite easy to check that $nm = (a^2+b^2)(c^2+d^2) = (ac-bd)^2+(ad+bc)^2$. But, this result can be appreciated much more if we observe that if n (and similarly m) is the sum of two squares, then n (and similarly m) is the modulus of a complex number (both real and imaginary parts being integers - such complex numbers are called Gaussian integers). Let $n = |z|$ and $m = |w|$, then $nm = |z||w| = |zw|$. In other words, nm is the modulus of the complex number zw. Hence, it is the sum of two squares, namely, the sum of the squares of the real and imaginary parts of the complex number. Thus, complex numbers help us understand this result in a much deeper sense. Now one may ask:

Question: Is the collection of natural numbers that can be written as the sum of three squares closed under multiplication?

The answer to this question turns out to be no. To see this, observe that $3 = 1^2 + 1^2 + 1^2$ and $5 = 0^2 + 1^2 + 2^2$. But, we claim $3 \times 5 = 15$ cannot be written as the sum of three squares. First of all notice that if $15 = a^2 + b^2 + c^2$, then $a, b, c \leq 3$ as $4^2 = 16 > 15$. Moreover, notice that if one of the three (a,b, or c) is equal to 3, then the sum of the other two squares would become $15 - 9 = 6$. But, this is not possible. So, $a, b, c \leq 2$. But, then $a^2 + b^2 + c^2 \leq 2^2 + 2^2 + 2^2 = 12 < 15$. Thus, we cannot find such a,b, and c.

Recall that we can define a norm on $\mathbb{R}^3$ quite analogous to how we defined a norm on $\mathbb{R}^2$. More precisely, we can define $|(x, y, z)| = x^2 + y^2 + z^2$. Thus, if we could define a multiplication on $\mathbb{R}^3$ such that

(1) If $v, w \in \mathbb{N}^3 \subset \mathbb{R}^3$, then $vw \in \mathbb{N}^3 \subset \mathbb{R}^3$
(2) $|vw| = |v||w|$

then, the answer to the second question should have been yes, just like the answer to the first question. Thus, it is clear that we cannot define a multiplication on $\mathbb{R}^3$ satisfying both conditions. But, what if we drop the first condition? One may ask:

Question: Can one define a multiplication on $\mathbb{R}^3$ such that $|vw| = |v||w|$?

Honestly, I do not know the answer to this question. I believe the answer is no, but do not know how to prove it. We certainly need the multiplication to satisfy some nice properties - as the answer to the following questions in yes.

Question: Can we define a field structure on $\mathbb{R}^3$?

This is because $\mathbb{R}^3$ has the same cardinality as $\mathbb{R}$. In other words, there is a bijection from $\mathbb{R}^3$ to $\mathbb{R}$. This bijection can be used to construct an addition and multiplication on $\mathbb{R}^3$ and that would turn $\mathbb{R}^3$ into a field isomorphic to $\mathbb{R}$.

The field structure on $\mathbb{R}^2$ (or $\mathbb{C}$) we constructed was somewhat special. We treat numbers of the form $a + 0i$ (or equivalently $(a, 0)$) are real numbers. The scalar multiplication by the real number a (when we treat $\mathbb{R}^2$ or $\mathbb{C}$ as a vector space) matches with multiplication (as in the field operation) by the "real number" $(a, 0)$. Thus, the vector space structure and the field structure are mutually compatible. When this happens we say it forms an algebra. If we demand all properties except commutativity of multiplication, we obtain a division algebra. The famous Frobenius theorem states that up to isomorphism there are only three finite dimensional division algebras - $\mathbb{R}$, $\mathbb{C}$ and Quaternions. And this theorem makes me believe that if the multiplication is "nice enough", the answer to Question "Can one define a multiplication on $\mathbb{R}^3$ such that $|vw| = |v||w|$?" should be no . The existence of Quaternions also implies that the answer to the following question is yes.

Question: Is the collection of natural numbers that can be written as the sum of four squares closed under multiplication?

And I encourage you to explore the most general question in this line

Question: Is the collection of natural numbers that can be written as the sum of n squares closed under multiplication?

Exercises

(1) Let $z = 2 + 3\imath$ and $w = 5 - 6\imath$. Compute

- (a) $z - w$
- (b) $z \times w$
- (c) $\frac{1}{z}$
- (d) $\frac{z}{w}$

(2) Check which of the properties of a field hold for addition and multiplication of natural numbers ($\mathbb{N}$), integers ($\mathbb{Z}$), and rational numbers ($\mathbb{Q}$).

(3) Let $\mathbb{Z}[x]$ denote polynomials with integer coefficients and $\mathbb{R}[x]$ denote polynomials with real coefficients. Check which of the properties of a field hold for addition and multiplication of elements in $\mathbb{Z}[x]$ and $\mathbb{R}[x]$.

(4) Show that $\mathbb{Z}/3\mathbb{Z}$ is a field.

(5) Show that $\mathbb{Z}/4\mathbb{Z}$ is not a field.

(6) Show that $\mathbb{Z}/n\mathbb{Z}$ is a field if and only if n is prime.

(7) Express $5 - 6\iota$ in polar coordinates.

(8) Represent the following graphically and justify your answers
   (a) $\{z \in \mathbb{C} : |z - \iota| = 1\}$
   (b) $\{z \in \mathbb{C} : |z - \iota| = |z - 1|\}$
   (c) $\{z \in \mathbb{C} : |z - (1 + \iota)| = |z|\}$
   (d) $\{z \in \mathbb{C} : arg(z) = \frac{\pi}{4}\}$
   (e) $\{z \in \mathbb{C} : arg(z - i) = arg(z - 1)\}$

(9) Compute $\|1 + i2\|_p$ for $p = 1, 2, 3, \infty$.

(10) Suppose $u, v \in \mathbb{R}^2$ are two vectors such that $\|u\| = 2$, $\|u + v\| = 3$, and $\|u - v\| = 1$. Find $\|v\|$.

(11) Give an example or prove the non-existence of a function $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ that is
   (a) symmetric and bilinear but not positive definite
   (b) bilinear and positive definite but not symmetric
   (c) symmetric and positive definite but not bilinear

(12) Give an example or prove the non-existence of a function $f : \mathbb{R}^2 \to \mathbb{R}$ that
   (a) is homogeneous and satisfies the triangle inequality but is not positive definite
   (b) is positive definite and satisfies the triangle inequality but is not homogeneous
   (c) is homogeneous and positive definite but does not satisfy the triangle inequality

(13) Give an example or prove the non-existence of a function $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ that
   (a) is symmetric and satisfies the triangle inequality but is not positive
   (b) is positive and satisfies the triangle inequality but is not symmetric
   (c) is positive and symmetric but does not satisfy the triangle inequality

(14) In class, we saw that given an inner product $\langle , \rangle$ on a vector space V, there is an associated norm. But, not every norm is induced by an inner product in this manner. Suppose we knew that $\|\|$ is induced by some inner product. Can you discover this inner product? More precisely, can you express $\langle u, v\rangle$ as some function of $\|u\|$, $\|v\|$ $\|u + v\|$, and $\|u - v\|$.

(15) Given an r, show that there exists an s such that

   (a) $B_{d_1}(x, s) \subset B_{d_2}(x, r)$
   (b) $B_{d_2}(x, s) \subset B_{d_1}(x, r)$
   (c) $B_{d_1}(x, s) \subset B_{d_\infty}(x, r)$
   (d) $B_{d_\infty}(x, s) \subset B_{d_1}(x, r)$
   (e) $B_{d_\infty}(x, s) \subset B_{d_2}(x, r)$
   (f) $B_{d_2}(x, s) \subset B_{d_\infty}(x, r)$

(16) Use the previous exercise to show that, if $p, q \in \{1, 2, \infty\}$, a set U is open in $(\mathbb{C}, d_p)$ iff U is open in $(\mathbb{C}, d_q)$.

(17) If $d(x_n, x) \to 0$ and $d(x_n, y) \to 0$, then show that $x = y$. That is, the limit of a sequence is unique.

(18) Show that, in the metric space $(\mathbb{C}, d_1)$, a sequence $z_n = x_n + iy_n$ converges to a point $z = x + iy$ iff x_n converges to x and y_n converges to y.

(19) Show that, in the metric space $(\mathbb{C}, d_\infty)$, a sequence $z_n = x_n + iy_n$ converges to a point $z = x + iy$ iff x_n converges to x and y_n converges to y.

(20) Show that in any metric space (X, d) a sequence x_n converges to a point x iff for every open set U in X, there exists an $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$.

(21) Use GeoGebra/Desmos to visualise the open balls in $(\mathbb{C}, d_p)$ where $p = 3, 4, 5$.

(22) Check the convergence of the following sequences:

- (a) i^n
- (b) $\frac{1}{n} + i\frac{-1}{n}$
- (c) $\frac{1}{n} + i(-1)^n$
- (d) $(1+i)^n$
- (e) $\frac{1}{n} + i(-1)^n$
- (f) $\frac{2^n}{n}$
- (g) z^n where $|z| < 1$
- (h) z^n where $|z| > 1$
- (i) z^n where $|z| = 1$, but $z \neq 1$

(23) Give an example of the following if they exist. Else, explain why such an example does not exist.
- (a) A sequence z_n such that $|z_n|$ converges, but z_n does not.
- (b) A sequence z_n such that z_n converges, but $|z_n|$ does not.
- (c) A sequence z_n such that $|z_n| < N$ for some $N \in \mathbb{N}$, but z_n does not converge.
- (d) A sequence z_n such that $|z_n| < N$ for some $N \in \mathbb{N}$, but z_n does not have any convergent subsequence.
- (e) A metric $d : X \times X \to \mathbb{R}$ such that $d(x, y)$ takes exactly 3 distinct values.
- (f) A metric on $\mathbb{C}$ where the open balls are ellipses but not circles.
- (g) A metric on $\mathbb{C}$ where the open balls are rectangles but not squares.
- (h) A metric on $\mathbb{C}$ where the open balls are parellelograms but not rectangles/squares.

(24) Show that arbitraty interstection of closed sets is closed. Show that finite union of closed sets is closed.

(25) Let $d_E : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the function $d_E(x, y) = |x - y|$. Show that d_E is a metric.

(26) Show that a subset U of $(\mathbb{R}, d_E)$ is open iff it is a **disjoint** union of open intervals. (Note that every open interval is an open ball)

(27) Consider the metric space $(\mathbb{R}^2, d_2)$. Clearly $\mathbb{R}^2$ is an open subset. Show that $\mathbb{R}^2$ cannot be written as a **disjoint** union of open balls. What happens if we consider $(\mathbb{R}^2, d_\infty)$ instead?

(28) Which of the following subsets of $\mathbb{C}$ are open? Which of the following subsets of $\mathbb{C}$ are closed? Prove your claims

- (i) $\mathbb{C}$
- (ii) $\mathbb{N}$
- (iii) $\{x + i0 : 0 \leq x \leq 2\}$
- (iv) $\{z \in \mathbb{C} : 2 < |z| < 3\}$
- (v) $\left\{x + i\sin\left(\frac{1}{x}\right) : x \in \mathbb{R}\right\}$
- (vi) $\{e^{in} : n \in \mathbb{N}\}$

(29) Find the real and imaginary parts of $\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)^{100}$.

(30) Is the collection of natural numbers that can be written as the sum of two cubes closed under multiplication?

Holomorphic functions

Last chapter gave us a thorough understanding of the various structures that can be equipped on $\mathbb{C}$. Analysis, however, is the study of functions, so we should have a rich class of functions $f : \mathbb{C} \to \mathbb{C}$ at our disposal. Interestingly, they are already at your disposal.

2.1. Two Recipes

Broadly there are two strategies to come up with functions $f : \mathbb{C} \to \mathbb{C}$ or from $f : D \to \mathbb{C}$ where D is a subset of $\mathbb{C}$. Historically, the word function was almost synonymous to an algebraic expression - sometimes also called a formula. The more abstract definitions of functions (think of the Dirichlet or Thomae functions) are a more recent phenomenon - meaning only a couple of centuries old. Thus, polynomials - expressions of the form $a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ - can be treated as functions. However, the algebraic expressions, our ancestors used were not always as simple. They also dealt with more complicated expressions which will be the content of our next chapter. Polynomials and their cousins use the structure of $\mathbb{C}$ to construct functions.

There is an easier way to construct functions $f : \mathbb{C} \to \mathbb{C}$. Notice that given a function $F : \mathbb{R}^2 \to \mathbb{R}^2$, it corresponds canonically to a function $f : \mathbb{C} \to \mathbb{C}$. More precisely, if $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ is the function $\pi_1(x, y) = x$ and $\pi_2 : \mathbb{R}^2 \to \mathbb{R}$ is the function $\pi_2(x, y) = y$,

$$f(x + iy) = \pi_1(F(x, y)) + i\pi_2(F(x, y)).$$

Similarly, given a function $f : \mathbb{C} \to \mathbb{C}$, it corresponds canonically to a function $F : \mathbb{R}^2 \to \mathbb{R}^2$. More precisely,

$$F(x, y) = (Re(f(x + iy)), Im(f(x + iy))).$$

Thus, the identification of $\mathbb{R}^2$ and $\mathbb{C}$ create an identification of the set of functions $F : \mathbb{R}^2 \to \mathbb{R}^2$ - $\mathcal{F}(\mathbb{R}^2, \mathbb{R}^2)$ - and the set of functions $f : \mathbb{C} \to \mathbb{C}$ - $\mathcal{F}(\mathbb{C}, \mathbb{C})$. Moreover, recall that a sequence $z_n = x_n + iy_n$ converges to $z = (x, y)$ iff (x_n, y_n) converges to (x, y). Thus,

Theorem 2.1. *A function $F : \mathbb{R}^2 \to \mathbb{R}^2$ is continuous iff the corresponding function $f : \mathbb{C} \to \mathbb{C}$ is continuous.*

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "F"] \arrow[d, leftrightarrow] & \mathbb{R}^2 \arrow[d, leftrightarrow] \\
\mathbb{C} \arrow[r, "f"] & \mathbb{C}
\end{tikzcd}$$

The one-one correspondence between the $\mathcal{F}(\mathbb{R}^2, \mathbb{R}^2)$ and $\mathcal{F}(\mathbb{C}, \mathbb{C})$ gives us plenty of examples of functions $F : \mathbb{C} \to \mathbb{C}$. Moreover, the above theorem helps us understand the continuity of functions $F : \mathbb{C} \to \mathbb{C}$.

Lemma 2.2. *If $z = x + iy$, then the real part and imaginary part of z^n are polynomials in x and y.*

Proof by Induction. The base case is pretty obvious. Assume that $z^k = p(x, y) + iq(x, y)$. Then $z^{k+1} = z^k.z = (p(x, y) + iq(x, y))(x + iy) = xp(x, y) - yq(x, y) + i(yp(x, y) + xq(x, y))$. From the expression, it is clear that the real and imaginary parts of z^{k+1} are also polynomials. Thus, by induction, we have the result. ☐

From the above lemma, the following result is straightforward

Lemma 2.3. *If $f : \mathbb{C} \to \mathbb{C}$ is a polynomial, then the corresponding function $F : \mathbb{R}^2 \to \mathbb{R}^2$ is both continuous and differentiable.*

Thus, given a non-differentiable function $F : \mathbb{R}^2 \to \mathbb{R}^2$, the corresponding function cannot be a polynomial. This observation can be used to construct examples of functions that are not polynomials.

Example 2.4. The function $f : \mathbb{C} \to \mathbb{C}$ defined as $f(x + iy) = |x| + i|y|$ is not differentiable and hence cannot be a polynomial.

Unfortunately, even if we restrict our attention to continuous and differentiable functions $f : \mathbb{R}^2 \to \mathbb{R}^2$, the corresponding functions $F : \mathbb{C} \to \mathbb{C}$ need not be a polynomial. And this follows from another regularity of polynomials. The fundamental theorem of algebra (which we will hopefully prove in this course) implies that

Lemma 2.5. *If $f : \mathbb{C} \to \mathbb{C}$ is a polynomial, then $f^{-1}(0)$ is a finite set.*

This allows us to construct several examples of functions that are continuous and differentiable but not polynomials.

Example 2.6. Let $f : \mathbb{C} \to \mathbb{C}$ be the function $f(z) = \sin(x) + i\cos(y)$. Then, the zero set is a countable set and hence f is not a polynomial.

Example 2.7. Let $f : \mathbb{C} \to \mathbb{C}$ be the function $f(z) = x$. Then, the zero set is an uncountable set and hence F is not a polynomial.

The previous example can be generalised to

Example 2.8. Let $f : R \to \mathbb{R}$ be any differentiable function with at least one zero. And let $F : \mathbb{C} \to \mathbb{C}$ be the function $F(x + iy) = f(x)$. Then the zero set is an uncountable set and hence F is not a polynomial

2.2. Cauchy-Riemann equations

Anyways, you will probably agree that polynomials are too nice, and therefore we should not stick to just polynomials. But, I claim, differentiable functions $f : \mathbb{R}^2 \to \mathbb{R}^2$ is also not the right class of functions. To understand this claim, let us recall that the derivative of a function f at a point a is a linear approximation of f near the point a. However, the notion of linearity is dependent on the underlying field. And I hope you would agree with me that you should probably look for linearity when $\mathbb{C}$ is treated as a vector space over $\mathbb{C}$ - else it is pretty much the same as $\mathbb{R}^2$.

Then, the functions $F : \mathbb{C} \to \mathbb{C}$ defined as $f(z) = \overline{z}$ corresponds to the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $f(x, y) = (x, -y)$. The derivative of this function at any point is the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The function defined by the above matrix namely $f(x, y) = (x, -y)$ is a linear function from $\mathbb{R}^2 \to \mathbb{R}^2$, but the corresponding function $f : \mathbb{C} \to \mathbb{C}$ defined as $f(z) = \overline{z}$ is not a linear map, as $f(i.i) = f(-1) = -1 \neq +1 = i.(-i) = if(i)$. In fact,

THEOREM 2.9. *Every linear map from $\mathbb{C}$ to $\mathbb{C}$ when viewed as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ is given by a matrix*

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

Proof. Recall that if $L : \mathbb{C} \to \mathbb{C}$ is linear, then $L(z) = L(z.1) = z.L(1) = L(1).z$. So, if we denote, $L(1)$ by $a+ib$, then $L(z) = (a+ib)z$. But, we saw earlier that this correspond to the linear map

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

□

Thus, given a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $f(x, y) = (u(x, y), v(x, y))$, its derivative given by the matrix

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}.$$

is $\mathbb{C}$-linear iff

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

The above equations are famously known as the Cauchy-Riemann equations.

I encourage you to take some polynomials and verify that they satisfy Cauchy-Riemann equations. In fact, it can be proved that every polynomial satisfies the Cauchy-Riemann equations. However, in these notes, we will not do so. We would rather obtain this fact as a consequence of the results in the next section. You are of course encouraged to prove it. It is very difficult to find function $f : \mathbb{C} \to \mathbb{C}$ that are not polynomials but satisfy Cauchy-Riemann equations. Next chapter we will explore such functions. But, we can already discuss examples on subsets of $\mathbb{C}$.

EXAMPLE 2.10. Consider the function, $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ defined as $f(z) = \frac{1}{z}$. Then, $f(x+iy) = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}$. Clearly, the function is differentiable when treated as a function $F : \mathbb{R}^2 \to \mathbb{R}^2$. Moreover,

$$\frac{\partial u}{\partial x} = \frac{y^2 - x^2}{(x^2+y^2)} = \frac{\partial v}{\partial y}$$

and

$$\frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2} = -\frac{\partial v}{\partial x}$$

More generally,

EXAMPLE 2.11. Let $p(z)$ and $q(z)$ be two polynomials and $Z(q)$ be the zero set of q. Then the function $f : \mathbb{C} \setminus Z(q) \to \mathbb{C}$ defined as $f(z) = \frac{p(z)}{q(z)}$ satisfies the Cauchy-Riemann equations. The proof of this fact is again a consequence of the results in the next section.

2.3. Holomorphicity

Recall that the major difficulty while moving from differentiation of functions $f : \mathbb{R} \to \mathbb{R}$ to differentiation of functions $f : \mathbb{R}^2 \to \mathbb{R}^2$ was the lack of multiplication/division in $\mathbb{R}^2$. However, this is not the case when it comes to $\mathbb{C}$. Thus, we could bypass the route through $\mathbb{R}^2$ and directly define differentiation of functions $f : \mathbb{C} \to \mathbb{C}$ verbatim the same as that for functions $f : \mathbb{R} \to \mathbb{R}$. More precisely, we can say a function $f : \mathbb{C} \to \mathbb{C}$ is differentiable at a point z_0 if the limit

$$\lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists. Moreover, the limit is called the derivative of the function at z_0 and will be denoted as $f'(z_0)$. Notice however that we are implicity assuming that $f(z_0 + h)$ is defined for all h, because otherwise the term inside the limit will not make sense. However, as we are taking the limit as h tends to 0 it is enough that $f(z_0 + h)$ makes sense for all small values of h. In other words, the domain of the function should be open. From now on we will assume the domain of our function is open even when we do not state it explicitly. With this clarity, let us define

Definition 2.12. Let U be an open set. A function $f : U \to \mathbb{C}$ is said to be complex-differentiable at a point $z_0 \in U$ if

$$\lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exist. Moreover, when the function is complex-differentiable at z_0, the above limit is called the derivative of f at z_0 and is represented as $f'(z_0)$.

Definition 2.13. Let U be an open set. A function $f : U \to \mathbb{C}$ is said to be complex-differentiable on U iff f is complex-differentiable at all points in U.

Unless there is a chance for confusion, we are going to use differentiable to mean complex-differentiable whenever we talk about a function $f : \mathbb{C} \to \mathbb{C}$.

Example 2.14. The function $f : \mathbb{C} \to \mathbb{C}$ defined as $f(z) = c$ for some complex number c is differentiable and its derivative is 0 everywhere.

Example 2.15. The function $f : \mathbb{C} \to \mathbb{C}$ defined as $f(z) = z^n$ is differentiable and its derivative $f'(z) = nz^{n-1}$.

$$\begin{aligned}
\lim_{h\to 0} \frac{1}{h}\left[(z+h)^n - z^n\right] &= \lim_{h\to 0} \frac{1}{h}\left[\left(\sum_{k=0}^{n} \binom{n}{k} z^k h^{n-k}\right) - z^n\right] \\
&= \lim_{h\to 0} \frac{1}{h}\left(\sum_{k=0}^{n-1} \binom{n}{k} z^k h^{n-k}\right) \\
&= \lim_{h\to 0} \left(\sum_{k=0}^{n-1} \frac{1}{h}\binom{n}{k} z^k h^{n-k}\right) \\
&= \sum_{k=0}^{n-1} \left(\lim_{h\to 0} \frac{1}{h}\binom{n}{k} z^k h^{n-k}\right) \\
&= \sum_{k=0}^{n-1} \left(\lim_{h\to 0} \binom{n}{k} z^k h^{n-k-1}\right)
\end{aligned}$$

Now,

$$\lim_{h\to 0}\binom{n}{k}z^k h^{n-k-1}$$

is non-zero iff $n-k-1=0$ iff $k=n-1$ and then $\binom{n}{k}=n$. Thus, we have the desired result.

EXAMPLE 2.16. The function $f(z)=\overline{z}$ is not differentiable.

$$\lim_{h\to 0}\frac{\overline{z+h}-\overline{z}}{h}=\lim_{h\to 0}\frac{\overline{h}}{h}$$

If $h_n=i/n$ then $\overline{h_n}/h_n$ tends to -1. On the other hand, if $h=1/n$, then $\overline{h_n}/h_n$ tends to 1. Thus, the above limit cannot exist.

Now, we can prove several theorems analogous to the results we have proved in real analysis.

THEOREM 2.17. *If f and g are differentiable at a point z, then $f+g$ is also differentiable at z and $(f+g)'(z)=f'(z)+g'(z)$*

Proof. As,

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}\text{ and }\lim_{h\to 0}\frac{g(z+h)-g(z)}{h}$$

exists, by the algebra of limits we know

$$\begin{aligned}
f'(z)+g'(z)&=\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}+\lim_{h\to 0}\frac{g(z+h)-g(z)}{h}\\
&=\lim_{h\to 0}\frac{f(z+h)-f(z)+g(z+h)-g(z)}{h}\\
&=\lim_{h\to 0}\frac{f(z+h)+g(z+h)-f(z)-g(z)}{h}\\
&=\lim_{h\to 0}\frac{f+g(z+h)-f+g(z)}{h}\\
&=(f+g)'(z)
\end{aligned}$$

□

EXERCISE 2.18. Show that if f is differentiable at z, then cf is also differentiable at z and $(cf)'(z)=cf'(z)$.

THEOREM 2.19. *If f and g are differentiable at a point z, then fg is also differentiable at z and $(fg)'(z)=f'(z)g(z)+f(z)g'(z)$*

Proof. Notice that,

$$\begin{aligned}
\frac{fg(z+h)-fg(z)}{h}&=\frac{f(z+h)g(z+h)-f(z)g(z)}{h}\\
&=\frac{f(z+h)g(z+h)-f(z+h)g(z)+f(z+h)g(z)-f(z)g(z)}{h}\\
&=\frac{f(z+h)g(z+h)-f(z+h)g(z)}{h}+\frac{f(z+h)g(z)-f(z)g(z)}{h}\\
&=f(z+h)\frac{g(z+h)-g(z)}{h}+\frac{f(z+h)-f(z)}{h}g(z)
\end{aligned}$$

Now, algebra of limits will fetch us the result. □

THEOREM 2.20. *If g is differentiable at a point z and $g(z)\neq 0$, then $1/g$ is differentiable at z and $(1/g)'(z)=-g'(z)/g(z)^2$.*

Proof. Notice that,

$$\begin{aligned}\frac{\frac{1}{g}(z+h)-\frac{1}{g}(z)}{h} &= \frac{\frac{1}{g(z+h)}-\frac{1}{g(z)}}{h}\\ &= \frac{g(z)-g(z+h)}{hg(z+h)g(z)}\\ &= \frac{g(z)-g(z+h)}{h}.\frac{1}{g(z+h)g(z)}\end{aligned}$$

Now, algebra of limits will fetch us the result. □

Exercise 2.21. Show that if f and g are differentiable at z and $g(z) \neq 0$, then f/g is differentiable and

$$\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z)-f(z)g'(z)}{g(z)^2}.$$

Theorem 2.22. *If f is differentiable at z and g is differentiable at $f(z) = w$. then $g \circ f$ is differentiable at z and $(g \circ f)'(z) = g'(f(z))f'(z)$.*

Proof. Notice,

$$\begin{aligned}\frac{g\circ f(z+h)-g\circ f(z)}{h} &= \frac{g(f(z+h))-g(f(z))}{h}\\ &= \frac{g(f(z+h))-g(f(z))}{h}.\frac{f(z+h)-f(z)}{f(z+h)-f(z)}\\ &= \frac{g(f(z+h))-g(f(z))}{f(z+h)-f(z)}.\frac{f(z+h)-f(z)}{h}\end{aligned}$$

Now, algebra of limits will fetch us the result. □

Theorem 2.23. *Let U be an open set and $f : U \to \mathbb{C}$ a function that is complex-differentiable at z_0. Then, f is continuous at the point z_0*

Proof. As f is differentiable at z_0, the following limit exists

$$\lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}.$$

We also know that

$$\lim_{h\to 0} h = 0.$$

Thus, by algebra of limits

$$\lim_{h=0} f(z_0+h)-f(z_0) = \lim h \to 0\frac{f(z_0+h)-f(z_0)}{h}\lim_{h\to 0} h = f'(z_0).0 = 0.$$

□

2.4. Holomorphicity and Cauchy-Riemann equations

Throughout the rest of this chapter, let U be an open subset of $\mathbb{C}$ - by abuse of notion, we will also consider this a subset of $\mathbb{R}^2$.

THEOREM 2.24. *If $f : U \to \mathbb{C}$ is complex-differentiable function, then f satisfies the Cauchy-Riemann equations. That is, if $f(x+iy) = u(x,y) + iv(x,y)$ where $u, v : \mathbb{R}^2 \to \mathbb{R}$, then*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Proof. The basic idea is to compare the limit as h tends to 0 along the real and imaginary axes. First, we will approach 0 along the real axis, that is, assume that $h = h + i0$, then

$$\begin{aligned} \frac{f(z+h)-f(z)}{h} &= \frac{u(x+h,y)+iv(x+h,y)-u(x,y)-iv(x,y)}{h} \\ &= \frac{u(x+h,y)-u(x,y)}{h} + i.\frac{v(x+h,y)-v(x,y)}{h} \end{aligned}$$

Now, we approach 0 along the imaginary axis, that is, take $h = 0 + ih$,

$$\begin{aligned} \frac{f(z+h)-f(z)}{ih} &= \frac{u(x,y+h)+iv(x,y+h)-u(x,y)-iv(x,y)}{ih} \\ &= \frac{u(x,y+h)-u(x,y)}{ih} + i.\frac{v(x,y+h)-v(x,y)}{ih} \\ &= \frac{v(x,y+h)-v(x,y)}{h} - i.\frac{u(x,y+h)-u(x,y)}{h} \end{aligned}$$

As f is differentiable, the limit as h tends to 0 in each of the above situations would be equal to $f'(z)$. Thus,

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

□

Or alternatively,

THEOREM 2.25. *Let U be an open set and $f : U \to \mathbb{C}$ be a function. We can write f as $F(x+iy) = u(x,y) + iv(x,y)$ where $u, v : \mathbb{R}^2 \to \mathbb{R}$. Define $F : \mathbb{R}^2 \to \mathbb{R}^2$ as $F(x,y) = (u(x,y), v(x,y))$. If f is complex differentiable at $z_0 = x_0 + iy_0$ then $F : \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable at (x_0, y_0) and satisfies the Cauchy-Riemann equations. More precisely, if $f'(z_0) = a + ib$, then*

$$DF_{(x_0,y_0)} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

Proof. Assume f is differentiable at z_0. Then,

$$\begin{aligned} f'(z_0) = \lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h} &\implies \lim_{h\to 0}\frac{f(z_0+h)-f(z_0)-hf'(z_0)}{h}=0 \\ &\implies \lim_{h\to 0}\left\|\frac{f(z_0+h)-f(z_0)-hf'(z_0)}{h}\right\|=0 \\ &\implies \lim_{h\to 0}\frac{\|f(z_0+h)-f(z_0)-hf'(z_0)\|}{\|h\|}=0 \\ &\implies \lim_{h\to 0}\frac{\|f(z+h)-f(z_0)-(h_1+ih_2)(a+ib)\|}{\|h\|}=0 \\ &\implies \lim_{h\to 0}\frac{\|f(z+h)-f(z_0)-(h_1a-h_2b)+i(h_1b+h_2a)\|}{\|h\|}=0 \\ &\implies \lim_{h\to(0,0)}\frac{\left\|F((x_0,y_0)+(h_1,h_2))-F(x,y)-\begin{bmatrix}a & -b\\ b & a\end{bmatrix}\begin{bmatrix}h_1\\ h_2\end{bmatrix}\right\|}{\|h\|}=0 \end{aligned}$$

This implies that F is differentiable at (x_0, y_0) with the derivative

$$DF_{(x_0,y_0)}=\begin{bmatrix}a & -b\\ b & a\end{bmatrix}.$$

□

THEOREM 2.26. *Let U be an open set and $f : U \to \mathbb{C}$ be a function. We can write f as $f(x+iy) = u(x, y) + iv(x, y)$ where $u, v : \mathbb{R}^2 \to \mathbb{R}$. Define $F : \mathbb{R}^2 \to \mathbb{R}^2$ as $F(x, y) = (u(x, y), v(x, y))$. If F is differentiable at (x_0, y_0) and satisfies the Cauchy-Riemann equations, then f is complex differentiable at $z_0 = x_0 + iy_0$.*

Proof. As F satisfies the Cauchy-Reimann equations at (x_0, y_0), the derivative at (x_0, y_0) will be of the form

$$DF_{(x_0,y_0)}=\begin{bmatrix}a & -b\\ b & a\end{bmatrix}.$$

We will prove that f is differentiable at z_0 and has the derivative $f'(z_0) = a + ib$. As, F is differentiable,

$$\lim_{h\to(0,0)}\frac{\left\|F((x_0,y_0)+(h_1,h_2))-F(x,y)-\begin{bmatrix}a & -b\\ b & a\end{bmatrix}\begin{bmatrix}h_1\\ h_2\end{bmatrix}\right\|}{\|h\|}=0$$

But, this is possible iff,

$$\begin{aligned} \lim_{h\to 0}\frac{\|f(z_0+h)-f(z_0)-(h_1a-h_2b)+i(h_1b+h_2a)\|}{\|h\|}=0 &\implies \lim_{h\to 0}\frac{\|f(z_0+h)-f(z_0)-h(a+ib)\|}{\|h\|}=0 \\ &\implies \lim_{h\to 0}\left\|\frac{f(z_0+h)-f(z_0)-h(a+ib)}{h}\right\|=0 \\ &\implies \lim_{h\to 0}\frac{f(z_0+h)-f(z_0)-h(a+ib)}{h}=0 \\ &\implies \lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}=(a+ib) \end{aligned}$$

Which implies can happen only if f is differentiable at z_0 and the derivative $f'(z_0) = a + ib$. □

Remark 2.27. The proofs of Theorem 2.25 and Theorem 2.26 illustrate the relation between the complex derivative and the total derivative of the corresponding function. On the other hand, Theorem 2.24 helps us understand that if the partial derivatives are continuous, then so is f'.

EXAMPLE 2.28. The function $f(z) = \begin{cases} 0 & \text{if } z = 0 \\ e^{-1/z^4} & \text{otherwise} \end{cases}$ obeys the Cauchy-Riemann equations everywhere, but is not continuous at 0 and hence not complex-differentiable at 0.

Remark 2.29. The Looman Menchoff theorem states that the only deviation from complex differentiablity is of the type described above, namely, where the function fails to be continuaous. More precisely, If $f : U \to \mathbb{C}$ is continuous and satisfies the Cauchy-Riemann equations, then it is necessarily complex differentiable on U. However, the proof of that theorem is beyond the scope of this course.

In practice, it is difficult to check whether $F : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable or $f : U \subset \mathbb{C} \to \mathbb{C}$ is complex differentiable. On the other hand, finding partial derivatives is easy. However, as seen in the example above, existence of partial derivatives does not guarantee continuity, let alone differentiability. However,

THEOREM 2.30. *Let $F : U \to \mathbb{R}^2$ be such that the partial derivatives of F exists everywhere on U and is continuous. Then, F is differentiable on U.*

Proof. If we write $F(x, y) = (u(x, y), v(x, y))$, then the function F is differentiable on U iff $u : U \to \mathbb{R}$ and $v : U \to \mathbb{R}$ are both differentiable. We will prove that u is differentiable and leave the other proof as an exercise. The main idea in the proof is a clever use of the mean value theorem. Also as the partial derivatives are continuous, we can choose some δ such that if $|x'| < \delta$ and $y' < \delta$ then,

$$\left\|\frac{\partial u}{\partial x}(x + x', y + y') - \frac{\partial u}{\partial x}(x, y)\right\| < \frac{\varepsilon}{2} \tag{2.1}$$

and

$$\left\|\frac{\partial u}{\partial y}(x + x', y + y') - \frac{\partial u}{\partial y}(x, y)\right\| < \frac{\varepsilon}{2}. \tag{2.2}$$

Choose h_i such that $|h_i| < \delta$. Then, observe that,

$$u(x + h_1, y + h_2) - u(x, y) = u(x + h_1, y + h_2) - u(x, y + h_2) + u(x, y + h_2) - u(x, y)$$

Applying mean value theorem to the continuously differentiable functions

(1) $f_1 : [0, 1] \to \mathbb{R}$ defined as $f_1(h) = u(x + \lambda h_1, y + h_2)$
(2) $f_2 : [0, 1] \to \mathbb{R}$ defined as $f_2(h) = u(x + h_1, y + \lambda h_2)$

we can find numbers $\lambda_i \in [0, 1]$ such that

$$u(x + h_1, y + h_2) - u(x, y + h_2) = h_1 \frac{\partial u}{\partial x}(x + \lambda_1 h_1, y + h_2)$$

and

$$u(x, y + h_2) - u(x, y) = h_2 \frac{\partial u}{\partial y}(x + h_1, y + \lambda_2 h_2).$$

That is,

$$\begin{aligned} u(x+h_1,y+h_2)-u(x,y) &= u(x+h_1,y+h+2)-u(x,y+h_2)+u(x,y+h_2)-u(x,y) \\ &= h_1\frac{\partial u}{\partial x}(x+\lambda_1 h_1,y+h_2)+h_2\frac{\partial u}{\partial y}(x+h_1,y+\lambda_2 h_2) \end{aligned}$$

Thus, the expression

$$\left\|u(x+h_1,y+h_2)-u(x,y)-\left(\frac{\partial u}{\partial x}(x,y),\frac{\partial u}{\partial y}(x,y)\right)\cdot(h_1,h_2)\right\|$$

can be simplified as

$$\left\|h_1\left[\frac{\partial u}{\partial x}(x+\lambda_1 h_1,y+h_2)-\frac{\partial u}{\partial x}(x,y)\right]+h_2\left[\frac{\partial u}{\partial y}(x+h_1,y+\lambda_2 h_2)-\frac{\partial u}{\partial y}(x,y)\right]\right\|.$$

By, triangle inequality, the above expression evaluates to a value less than or equal to

$$|h_1|\left\|\frac{\partial u}{\partial x}(x+\lambda_1 h_1,y+h_2)-\frac{\partial u}{\partial x}(x,y)\right\|+|h_2|\left\|\frac{\partial u}{\partial y}(x+h_1,y+\lambda_2 h_2)-\frac{\partial u}{\partial y}(x,y)\right\|.$$

From Equation 2.1 and Equation 2.2, the above value is less than or equalt to

$$\frac{\varepsilon}{2}|h_1|+\frac{\varepsilon}{2}|h_2|\le\frac{\varepsilon}{2}\|(h_1,h_2)\|+\frac{\varepsilon}{2}\|(h_1,h_2)\|\le\|h\|\varepsilon$$

Thus, if $\|h\|<\delta$, then $|h_i|<\delta$ and hence

$$\left\|u(x+h_1,y+h_2)-u(x,y)-\left(\frac{\partial u}{\partial x}(x,y),\frac{\partial u}{\partial y}(x,y)\right)\cdot(h_1,h_2)\right\|\le\|h\|\varepsilon.$$

Thus,

$$\frac{\left\|u(x+h_1,y+h_2)-u(x,y)-\left(\frac{\partial u}{\partial x}(x,y),\frac{\partial u}{\partial y}(x,y)\right)\cdot(h_1,h_2)\right\|}{\|h\|}\le\varepsilon.$$

□

From the proof of Theorem 2.24 it is clear that if the partial derivatives are continuous, so is f'. Thus, from now on, we will assume that our function is continuously differentiable on its domain.

2.5. Domains

We will now try to prove the analogue of another important theorem. Recall that if we have a function $f:(a,b)\to\mathbb{R}$ such that $f'(x)=0$ for all $x\in(a,b)$, then f is a constant. We would like to know for what open subsets U of $\mathbb{C}$ do we have an analogous result for functions $f:U\to\mathbb{C}$. Of course, openness is not enough, as illustrated by the function $f:(0,1)\cup(2,3)\to\mathbb{R}$ defined as $f(x)=0$ if $x\in(0,1)$ and $f(x)=1$ if $x\in(2,3)$. Clearly, the issue in the above example is that the domain is "broken" or not connected. This notion can be made precise.

Definition 2.31. We say $P\subset\mathbb{C}$ is path-connected, if given any two points p and q there exists a continuous curve $\gamma:[a,b]\to\mathbb{C}$ such that $\gamma([a,b])\subset P$ and $\gamma(a)=p$ and $\gamma(b)=q$.

Although in the definition we are only assuming that any two points can be connected using a continuous curve, actually we can do much better. We can show that we can connect any two points in an open path-connected set with a piecewise smooth curve.

Definition 2.32. A continuous curve $\gamma:[a,b]\to\mathbb{R}^2$ is said to be piecewise smooth if there exists $a=t_0<t_1<\cdots<t_n=b$ such that $\gamma|_[t_i,t_{i+1}]$ is smooth.

We can in fact also prove that we can smoothen the finitely many corners and show that any two points in an open path-connected set can be connected by smooth curves. But, that is a bit too technical for our course. To prove even the weaker result, we need a bit of machinery which will be introduced now.

Definition 2.33. Let $D \subset \mathbb{R}^n$ and $X \subset D$. We say X is an open subset of D if given any $x \in X$, there exists an ε such that $B(x, \varepsilon) \cap D \subset X$. A subset X is said to be a closed subset of D if its complement is an open subset of D.

Exercise 2.34. Show that the only subsets of $[a, b]$ that are both open and closed are the $\emptyset$ and $[a, b]$.

Exercise 2.35. Show that the only subsets of $[0, 1] \cup [2, 3]$ that are both open and closed are $\emptyset$, $[0, 1]$, $[2, 3]$, and $[0, 1] \cup [2, 3]$.

Theorem 2.36. *Let P be a path-connected space. Then, the only subsets of P that are both open and closed are $\emptyset$ and P.*

Proof. We will use proof by contradiction. Assume there is a proper non-empty subset X of P that is both a closed and open subset of P. Then its complement $Y = P \setminus X$ is also proper, non-empty, both open and closed. Consider a curve $\gamma : [a, b] \to P$ that connects a point $x \in X$ and a point $y \in Y$. As X is a non-empty open set, $A := \gamma^{-1}(X)$ is a non-empty open subset of $[a, b]$ (the ε-δ definition implies that the inverse image of open sets under a continuous function is open). Similarly, as Y is a non-empty open set $B := \gamma^{-1}(Y)$ is a non-empty open subset of $[a, b]$. But, $[a, b] = \gamma^{-1}(P) = \gamma^{-1}(X) \cup \gamma^{-1}(Y) = A \cup B$. As X and Y are disjoint, their inverses A and B also must be disjoint. Thus, $B = [a, b] \setminus A$ and as B is an open subset of $[a, b]$ A is a closed subset of $[a, b]$. Which makes A non-empty, proper subset of $[a, b]$ that is both an open and closed subset of $[a, b]$ contradicting what you proved in an earlier exercise. □

Exercise 2.37. Let P be some subset of $\mathbb{R}^2$. Let $\sim$ be defined as $x \sim y$ if there is a piecewise smooth curve connecting x and y. Show that $\sim$ is an equivalence relation.

Theorem 2.38. *Let $D \subset \mathbb{R}^2$ be open and path-conneceted. Fix an arbitrary point $p \in D$ and define $X := \{q \in D : q \sim p\}$. Then, $X = D$. Intuitively, this should be read as any two points in a domain set can be connected by a piecewise smooth curve.*

Proof. We will show that X is both an open and closed subset of P and hence $X = D$. Let $x \in X$, then notice that there is some ε such that $B(x, \varepsilon) \subset D$. Clearly, every point in $B(x, \varepsilon)$ can be connected to x by a straight line. That is if $y \in B(x, \varepsilon)$, then $y \sim x$. But, we know $x \sim p$. Therefore, by transitivity of $\sim$, $y \sim p$ and hence $y \in X$. Thus, $B(x, \varepsilon) \subset X$ and as $x \in X$ was arbitrary X is open. We will now show that $Y = D \setminus X$ is also open. Let $y \in Y$ be arbitrary. As D is open, there exists an ε such that $B(y, \varepsilon) \subset D$. As before if $z \in B(y, \varepsilon)$, then $y \sim z$. If $z \sim p$, then transitivity of $\sim$ implies $y \sim p$, thus, $z \not\sim p$. Which means Y is open and hence X is both open and closed. Therefore $X = \emptyset$ or $X = D$. $X \neq \emptyset$ as $p \in X$. Hence proved. □

Theorem 2.39. *Let D be open and path-connected and $f : D \to \mathbb{C}$ be differentiable on D. If f is differentiable and $f'(z) = 0$ for all $z \in D$, then f is a constant.*

Remark 2.40. The proof of this theorem is an application of the following theorem you might have learnt in multi-variable calculus to the two functions u and v where $f = u + \iota v$.

Theorem 2.41. *Let D be an open path-connected subset of $\mathbb{R}^2$. Let $f : D \to \mathbb{R}$ be a continuously differentiable function such that $\nabla f = 0$ on D Then, f is a constant.*

Proof. We will use proof by contradiction. Assume, the function f is not constant. Thus, there would at least exist two points (x_0, y_0) and (x_1, y_1) in D such that $f(x_0, y_0) \neq f(x_1, y_1)$. As D is open and path-connected, we can connect the two points using a piecewise smooth curve $\gamma : [0, 1] \to \mathbb{R}^2$. More precisely, $\gamma(0) = (x_0, y_0)$, $\gamma(1) = (x_1, y_1)$, and $\gamma([0, 1]) \subset D$. The function $\phi = f \circ \gamma$ is differentiable on each of the intervals $[t_i, t_{i+1}]$ and by chain rule its derivative is identically 0 in the interval. Thus, we know that ϕ is a constant on the interval $[t_i, t_{i+1}]$ from the familiar result in the one-variable case. Thus, $f(x_0, y_0) = \phi(t_0) = \phi(t_1) = \cdots = \phi(t_n) = f(x_1, y_1)$. Which contradicts our assumption that $f(x_0, y_0) \neq f(x_1, y_1)$. ☐

Proof of Theorem 2.39. Let $f = u + \iota v$. If $f'(z) = 0$, then

$$\frac{\partial u}{\partial x} = 0 = \frac{\partial u}{\partial y} \text{ and } \frac{\partial v}{\partial x} = 0 = \frac{\partial v}{\partial y}.$$

Thus, applying the previous theorem to u and v, we know u and v are constant. Thus, f is also constant. ☐

As we want like to conclude the function is constant if its derivative is 0, we will assume that the domains of our functions are both open and path-connected. Thus, being not creative at all, we define

Definition 2.42. We say $D \subset \mathbb{C}$ is called a domain if it is both open and connected.

Theorem 2.43. *Let D be a domain and $f : D \to \mathbb{C}$ be differentiable. If f is real valued, then f is constant.*

Proof. Let $f = u + \iota v$. We see v is identically zero. From the Cauchy-Riemann equations, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0 \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial y} = 0.$$

Thus, u is a constant. ☐

Exercises

(1) Which of the following functions $f : \mathbb{C} \to \mathbb{C}$ are complex differentiable. Which of them are differentiable when treated as a function from from $\mathbb{R}^2$ to $\mathbb{R}^2$? Which of them are induced by a polynomial in z? Justify your claims.

(i) $f(z) = |z|$
(ii) $f(z) = (x^2 + y^2) - i(2xy)$
(iii) $f(x + iy) = (x^3 - 3xy^2) + i(3x^2y - y^3)$
(iv) $f(x + iy) = 2xy - i(x^2 - y^2)$

(2) Without appealing to their holomorphicity, show directly that every polynomial satisfies Cauchy-Riemann equations.

(3) Show that the function $f(x + \iota y) = \sqrt{|xy|}$ satisfies the Cauchy Riemann equations at $z = 0$, but is not differentiable at 0.
Remark: This does not contradict the Looman Menchoff theorem as although the partial derivatives are defined at 0, you cannot find a neigbourhod of 0 where the partial derivatives are well-defined!

(4) Show that $F : C \to \mathbb{C}$ is a complex-linear map iff the corresponding map $f : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies the following properties:

   (a) the matrix M_f of f has positive determinant.
   (b) $f(1,0)$ is orthogonal to $f(0,1)$
   (c) $\|f(1,0)\| = \|f(0,1)\|$

(5) Show that a linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$ preserve inner product, that is $\langle f(v), f(w)\rangle = \langle v, w\rangle$, iff
   (a) $f(1,0)$ is orthogonal to $f(0,1)$
   (b) $\|f(1,0)\| = \|f(0,1)\|$

(6) Let D be an open subset of $\mathbb{R}^2$ and $f : D \to \mathbb{R}^2$ be a differentiable map that satisfies the Cauchy-Riemann equations. Then, given any two curves $\alpha : [0,1] \to D$ and $\beta : [0,1] \to D$, $\langle \alpha'(t), \beta'(t)\rangle = \langle (f\circ\alpha)'(t), (f\circ\beta)'(t)\rangle$. Intuitively it is saying that the map "preserves angles". **Such maps are called conformal maps and was the main focus of Riemann.**

(7) Show that every rational function $f(z) = \frac{p(z)}{q(z)}$ is holomorphic on $\mathbb{C}\setminus Z(q)$ where $Z(q)$ represents the zero set of q. Thus, f satisfies the Cauchy-Riemann equations and is a conformal map on $\mathbb{C}\setminus Z(q)$.

(8) Show that $\mathbb{Q}$ is disconnected.

(9) Let $X = \{(x,y) : x^2 + y^2 \leq 1\} \cup \{(x,0) : x \in [1,2]\} \cup \{(x,y) : (x-3)^2 + y^2 \leq 1\}$. Show that X is path connected

(10) Let $f : (X,d) \to (Y,D)$ be a continuous function. Prove that if X is connected, then $f(X)$ is also connected. Prove that if X is path-connected, then $f(X)$ is also path-connected.

(11) Show that if U is an open connected subset of $\mathbb{C}$, then U is path connected.

(12) Let $f : [a,b] \to \mathbb{R}$ be a function. Show that if the graph of f is connected then f satisfies the intermediate value property.

(13) Let $f : \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) = \begin{cases} \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{otherwise} \end{cases}.$$

Show that the graph of f is connected, but not path connected. **This gives us an example of a connected space that is not path-connected.**

(14) Let D be a domain. Show that if $f : D \to \mathbb{C}$ and $\overline{f} : D \to \mathbb{C}$ are both differentiable, then f is a constant function.

(15) Let D be a domain. Show that if $f : D \to \mathbb{C}$ is differentiable and $|f|$ is constant, then f is a constant function.

(16) Let D be a domain and $f : D \to \mathbb{C}$ be continuously differentiable. Use the inverse function theorem from multi-variable calculus to prove that given any point $z_0 \in D$, there exists a neighbourhood $B(z_0, \varepsilon)$ such that $f|_{B(z_0,\varepsilon)}$ is invertible.

Power series

Recall we had discussed that polynomials form a very small class of functions and thus we need to enlarge our class of functions. Note that a polynomial $3x^2+5x+2$ can at once be thought of as an expression with no meaning or as a function defined for all real or complex values. If we instead look at an expression $\frac{x}{(x-1)(x-2)}$, it can be evaluated only when $x \neq 1$ and $x \neq 2$. So, the expression represents a function, but we have to restrict its domain. As a series is a much more complicated expression, finding the domain of the corresponding function requires some effort. So, we will first treat the series only as formal expression - with no meaning attached to the expression. After proving some properties of formal power series, we will look at a natural domain where we can treat it as a function.

3.1. Formal Power Series

DEFINITION 3.1. A formal power series is an expression of the form $\sum_{n=0}^{\infty} a_n z^n$. When it does not lead to any ambiguity, we might also write it as $\sum a_n z^n$.

Let us look at some important examples of power series:

(1) $\sum z^n$
(2) $e(z) = \sum \frac{z^n}{n!}$
(3) $\sin(z) = \sum \frac{(-1)^n}{(2n+1)!} z^{2n+1}$
(4) $\cos(z) = \sum \frac{(-1)^n}{(2n!)} z^{2n}$
(5) Let α be a non-zero complex number. Define the binomial coefficients

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+1)}{n!}$$

and the binomial series

$$B_\alpha(z) = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n.$$

Note that, by convention $\binom{\alpha}{0} = 1$.

Remark 3.2. Of course, we do not associate any meaning or treat the above as functions as of now. We are giving names only because they remind us of the familiar power series expansion for sin, cos etc.

Given two power series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$, we can add the two as follows:

$$\sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} (a_n + b_n) z^n.$$

Exercise 3.3. Prove that the collection of all power series form an abelian group under addition. More precisely, show that addition is associative, commutative, has an additive identity, every element has an additive inverse.

Hint: The power series $\sum_{n=0}^{\infty} 0.z^n$ is the additive identity and the inverse of $\sum_{n=0}^{\infty} a_n z^n$ is the power series $\sum_{n=0}^{\infty} (-a_n) z^n$.

We can also multiply the two power series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ as follows

$$\left(\sum_{n=0}^{\infty} a_n z^n\right) \times \left(\sum_{n=0}^{\infty} b_n z^n\right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k}\right) z^n.$$

Exercise 3.4. Prove that The collection of all power series form a ring under addition and multiplication. More precisely, show that multiplication is associative, has a multiplicative identity, and distributes over addition.

Hint: The power series $1 + \sum_{n=1}^{\infty} 0.z^n$ is the multiplicative identity.

Given a power series $\sum a^n z^n$ and a complex number α, we can define the scalar multiplication by *alpha* as $\alpha.(\sum a^n z^n) = \sum \alpha.a_n z^n$. This scalar multiplication along with addition defined earlier gives the collection of power series a vector space structure. Moreover, multiplication behaves well with the vector space structure and makes it an algebra.

Exercise 3.5. Show that the collection of power series equipped with the addition, multiplication, and scalar multiplication defined above forms an algebra. More precisely, show that multiplication is a bilinear operation, that is,

(1) $(f+g)h = fh + gh$
(2) $h(f+g) = hf + hg$
(3) $(\alpha.f)(\beta.g) = (\alpha\beta).fg$

Exercise 3.6. Show that the ring (algebra) of polynomials form a sub-ring (sub-algebra) of the ring (algebra) of power series.

Definition 3.7 (Units). Elements in a ring that have a multiplicative inverse are called unit.

Lemma 3.8. *The only units in the ring of polynomials are constant polynomials*

Proof. The easiest way to prove it is by using the degree of a polynomial, we will denote degree of a polynomial p Given two polynomials p and q, note that $\deg(pq) = \deg(p) + \deg(q)$. If p is a non-constant polynomial, then $\deg(p) \geq 1$. Thus, given any polynomial q, $\deg(pq) \geq 1$ and therefore pq cannot be equal to 1. ☐

This proof technique is really central to a lot of mathematics and will also allow us answer the analogous question for the ring of formal power series. What are the units in the ring of power series. One may be tempted to think the answer would still be the same. However, we already know that

$$(1-z)\sum z^n = 1 + (1-1)z + (1-1)z^2 + ... = 1.$$

And thus, the power series $1-z$ and $\sum z^n$ are both units. In fact, this idea can be used to prove an even more general result.

Lemma 3.9. *Any power series $f = \sum a_n z^n$ such that $a_0 = 1$ has a multiplicative inverse.*

Proof. The basic idea behind the proof is the observation that

$$(1-z)\sum z^n = 1 + (1-1)z + (1-1)z^2 + ... = 1.$$

Notice we may write $f(z) = 1 - h(z)$ where $h(z) = -\sum_{n=1}^{\infty} a_n z^n$. After the earlier observation, it is natural to guess that the inverse of f should be $\phi(z) = 1 + h(z) + h(z)^2 + \dots$ if it makes sense. More precisely, the coefficient of z^n should be a finite quantity. Notice that as $ord(h) \geq 1$, $ord(h^{n+1}) > n$. Thus, the coefficient of z^n in f is the same as the coefficient of z^n in $1 + h(z) + \cdots + h(z)^n$. ☐

Theorem 3.10. *Let f be a power series. If $ord(f) = 0$, then f has a multiplicative inverse.*

Proof. Let $f = \sum b^n z^n$ and we are given $b_0 \neq 0$. Thus, consider the power series $g = \frac{1}{b_0} f = \frac{1}{b_0} \sum b_n z^n = \sum \frac{b_n}{b_0} z^n$. By previous lemma g has a multiplicative inverse, which we will denote as $\frac{1}{g}$. Then $b_0.(\frac{1}{g})$ is the inverse of f, by the bilinearity of multiplication

$$f.\frac{b_0}{g} = \left(\frac{1}{b_0}.g\right)\left(b_0.\frac{1}{g}\right) = \left(\frac{1}{b_0} b_0\right) . \left(g.\frac{1}{g}\right) = 1.$$

☐

The idea of introducing an invariant comes handy when we try to prove the converse: If f has a multiplicative inverse, then $orf(f) = 0$.

Definition 3.11. Given a power series $f = \sum a_n z^n$, the smallest n such that $a_n \neq 0$ is called the order of the power series. We may denote the order of f as $ord(f)$.

Exercise 3.12. Let f and g be two power series. Show that $ord(fg) = ord(f) + ord(g)$.

Exercise 3.13. Let f be a power series. Show that if $ord(f) > 0$, then f does not have a multiplicative inverse.

Example 3.14. Let $f(z) = z^2 + 3$. Then,

$$\begin{aligned} \frac{1}{f(z)} &= \frac{1}{3} \frac{1}{1-(-z^2)} \\ &= \frac{1}{3}\left(1 + (-z^2) + (-z^2)^2 + (-z^2)^3 + \dots\right) \\ &= \frac{1}{3}\left(1 - z^2 + z^4 - z^6 + \dots\right). \end{aligned}$$

Example 3.15. Recall that $\cos(z) = \sum \frac{(-1)^n}{2n!} z^{2n}$. Thus,

$$\begin{aligned} \frac{1}{\cos(z)} &= \frac{1}{1 - \left(\frac{z^2}{2!} - \frac{z^4}{4!} + \dots\right)} \\ &= 1 + \left(\frac{z^2}{2!} - \frac{z^4}{4!} + \dots\right) + \left(\frac{z^2}{2!} - \frac{z^4}{4!} + \dots\right)^2 + \dots \\ &= 1 + \frac{z^2}{2!} + \left(\frac{1}{4} - \frac{1}{24}\right) z^4 + \text{higher order terms} \end{aligned}$$

This technique can sometime be used to find inverse of power series that do not have a non-zero constant term

Example 3.16. Recall that $\sin(z) = \sum \frac{(-1)^n}{(2n+1)!} z^{2n+1} = z(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots)$. Thus,

$$\begin{aligned}
\frac{1}{\sin(z)} &= \frac{1}{z(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots)} \\
&= \frac{1}{z}\left(1 + \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \dots\right) + \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \dots\right)^2 + dots\right) \\
&= \frac{1}{z} + \frac{z}{3!} - \left(\left(\frac{1}{3!}\right)^2 - \frac{1}{5!}\right) z^3 + \text{higher order terms.}
\end{aligned}$$

However, this is not a power series as there is a negative power of z. Allowing finitely many negative powers is useful and such expressions are well studied. But, let us not spend much time on it now.

3.2. Convergence of a series of real numbers

Last section, we discussed formal expressions called power series and studied some of its properties. Of course, the motivation was to eventually treat them as functions on suitable domains. To this end, we are interested in the question: For what values of z does the series of complex numbers $\sum a_n z^n$ make sense? Before we begin the study of series of complex numbers, let us make sure we are comfortable with the basics of series of real numbers.

A series $\sum_{n=0}^{\infty} x_n$ is an infinite sum. Let us first look at an example known as the alternating series. This is the series $\sum (-1)^n$. Notice that the following two groupings give rise to two different answers.

$$(1 + (-1)) + (1 + (-1)) + (1 + (-1)) + \dots = 0 + 0 + 0 + \dots = 0$$

while

$$1 + ((-1) + 1) + ((-1) + 1) + ((-1) + 1) + \dots = 1 + 0 + 0 + 0 + \dots = 1.$$

Thus, unless we make sense of an infinite sum in a more sophisticated manner, we cannot expect associativity to follow for infinite sums. And the definition of a sum of a series you are familiar with is already good enough.

Definition 3.17. Given a series of real number $\sum_{n=0}^{\infty} x_n$, we say the series converge, if the sequence of partial sums $s_k = \sum_{n=0}^{k} x_n$ converge as k tends to infinity. And the limit is called the sum of the series.

Exercise 3.18. Show that the partial sums for the alternating sequence s_n is 1 when n is odd and 0 when n is even. Hence, the series does not converge.

Thus, we have excluded the problematic example and hopefully infinite sum would be associative whenever the series converge. And this is indeed the case.

Theorem 3.19. *If $\sum a_n$ is a convergent series of real numbers, then the sum does not depend on how you group the numbers. More precisely, if $0 = i_0 < i_1 < i_2 < \dots$ be an increasing sequence of natural numbers, then*

$$\sum_{k=1}^{\infty} \left(\sum_{i=i_{k-1}}^{i_k - 1} a_n \right)$$

converge and

$$\sum_{k=1}^{\infty} \left(\sum_{i=i_{k-1}}^{i_k - 1} a_n \right) = \sum_{n=0}^{\infty} a_n.$$

Proof. Let s_j be the partial sums of $\sum a_n$ and S_j be the partial sums of

$$\sum_{k=1}^{\infty}\left(\sum_{i=i_{k-1}}^{i_k-1} a_n\right).$$

Then, note that

$$S_j = \sum_{k=1}^{j}\left(\sum_{i=i_{k-1}}^{i_k-1} a_n\right) = s_{i_j-1}.$$

Thus, the sequence S_j is a subsequence of the sequence s_j. Therefore S_j converge and has the same limit as s_j. $\square$

Thus, as long as the series converges, they behave not so differently from finite sums. Hence, we will study infinite sums only when the series converge. Another important example of series is the geometric series.

Example 3.20 (Geometric series). The geometric series is a series of the form $\sum x^n$. Then the sequence of partial sum $s_n = \sum_{i=0}^{n} x^i$. Notice that $s_n - xs_n = 1 + x + \ldots x^n - (x + x^2 + \cdots + x^{n+1}) = 1 - x^{n+1}$. It is easy to see that the series does not converge when $x = 1$ as then $s_n = n$. So, assume $x \neq 1$. Thus,

$$s_n = \frac{1 - x^{n+1}}{1 - x}.$$

From the properties of limits, it is clear that s_n converges iff x^{n+1} converges. x^{n+1} converges iff $x \in (-1, 1]$. But, we have already assumed that $x \neq 1$, thus s_n converges iff $x \in (-1, 1)$.

In the above example, we noticed that the series converges only when the $n - th$ summand converges. And this is something we should expect. If the modulus of the summands are bounded below, then we should expect the sum to blow up. This intuition is captured in the following theorem.

Theorem 3.21. *If* $\sum x_n$ *converges then* x_n *converges to* 0.

Proof. We know the series $\sum x_n$ converges iff the sequence of partial sums s_n converge. But, notice that $x_n = s_n - s_{n-1}$. As s_n and s_{n-1} converge, so does their difference x_n. Moreover, $\lim x_n = \lim s_n - \lim s_{n-1} = \sum x_n - \sum x_n = 0$. $\square$

The converse of this theorem is however not true as demonstrated by the following example. Intuitively, this says, it is not enough that x_n converge to 0, but it should converge fast enough.

Example 3.22 (Harmonic series). The harmonic series is the series of the form $\sum_{n=1}^{\infty} \frac{1}{n}$. Notice that as the summands are all positive, the sequence of partial sums form and increasing sequence. Hence, the sequence of partial sums converge iff it is bounded. We will show that as s_n is not bounded above the series is divergent that means not convergent.

First observe that, as $\frac{1}{3} > \frac{1}{4}$, $\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. Similarly, $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$. And this process can be repeated. So, in some sense we can get infinitely many halves if we proceed and because we can find infinitely many halves the sum of them cannot be bounded. That's the basic idea but we will just make it a little more precise.

We will prove using mathematical induction that $s_{2^n} > 1 + \frac{n}{2}$. This is obvious for $n = 1$ as $s_2 = 1 + \frac{1}{2}$. Asssume it is true for k, that is, assume that $s_{2^k} > 1 + \frac{k}{2}$. Then,

$$s_{2^{k+1}} = s_{2^k} + \frac{1}{2^k + 1} + \cdots + \frac{1}{2^{k+1}} > 1 + \frac{k}{2} + \frac{1}{2^{k+1}} + \cdots + \frac{1}{2^{k+1}} = 1 + \frac{k}{2} + \frac{1}{2} = 1 + \frac{k+1}{2}.$$

Hence we have the claim by the principle of mathematical induction. Thus, we clearly see that the sequence of partial sums is not bounded and hence does not converge.

This implies that

Example 3.23 (Harmonic p series, $0 < p \leq 1$). The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges for $0 < p \leq 1$. Notice that as $p < 1$, $n^p < n$. This implies $\frac{1}{n^p} > \frac{1}{n}$. Thus the sequence of partial sums of $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is bounded below by the sequence of partial sums of $\sum_{n=1}^{\infty} \frac{1}{n}$. Hence, the sequence of partial sums certainly cannot be bounded above. Hence they do not converge.

The idea we have used in the above example is an illustration of a more general principle stated in the theorem below.

Theorem 3.24 (Comparison theorem). *Let $\sum a_n$ and $\sum b_n$ be two series such that $0 < a_n < b_n$ for all n. Then, if $\sum b_n$ converges, so does $\sum a_n$.*

Proof. Notice that as a_n and b_n are positive, the partial sums of $\sum a_n$ (say s_n) and partial sums of b_n (say t_n) are both increasing. Thus, the sequence of partial sums converge iff they are bounded above. Further, clearly, $s_n \leq t_n$. Thus, if t_n is bounded above, so is s_n. ☐

Remark 3.25. Under same hypothesis, if $\sum a_n$ diverges, we may thus conclude $\sum b_n$ diverges.

Example 3.26 (Harmonic 2 series). The harmonic 2 series is the series of the form $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Once again, the summands are all positive and hence the sequence of partial sums is monotonically increasing. Thus, the sequence of partial sums converge iff it is bounded above. We will prove that the sequence of partial sums is indeed bounded in this case.

Notice that $\frac{1}{2^2} + \frac{1}{3^2} < \frac{1}{2^2} + \frac{1}{2^2} = \frac{1}{2}$. Similarly, $\frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} < \frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{4^2} = \frac{1}{4}$. More generally, you can prove using induction (and this time I leave it as an exercise) that $s_{2^{n+1}-1} < 1 + \frac{1}{2} + \dots \frac{1}{2^n}$. Now, the left hand side is bounded above by the sum of the geometric series where $x = \frac{1}{2}$.

More generally (using very similar argument) one can prove that

Exercise 3.27 (Harmonic p series. $p > 1$). Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge for all $p > 1$.

Example 3.28 (Alternating harmonic series). The alternating harmonic series is the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

As

$$s_{2n} - s_{2(n-1)} = \frac{1}{2n-1} - \frac{1}{2n} \geq 0$$

the subsequence s_{2n} is monotonically increasing. Similarly,

$$s_{2n+1} - s_{2(n-1)+1} = \frac{1}{2n+1} - \frac{1}{2n} \leq 0$$

and thus **the subsequence s_{2n+1} is monotonically decreasing**. Further, notice that $s_2 \geq 0$ and s_{2n} is increasing implies that $s_{2n} \geq 0$ for all n. Similarly, $s_1 \leq 1$ and s_{2n+1} is decreasing implies that $2_{2n+1} \leq 1$ for all n. Finally, notice that $0 \leq s_{2n} \leq s_{2n+1} \leq 1$. Thus, s_{2n} is monotonically increasing and bounded above, hence s_{2n} converges. Similarly, s_{2n+1} is monotonically decreasing and bounded below, hence s_{2n+1} converges. Moreover,

$$\lim_{n\to\infty} (s_{2n+1} - s_{2n}) = \lim_{n\to\infty} \frac{1}{2n+1} = 0.$$

Thus, s_{2n+1} **and** s_{2n} **converge to the same limit**. Moreover, as s_{2n} and s_{2n+1} are bounded between 0 and 1, s_n is a bounded sequence. Recall that a bounded sequence diverges (does not converge) iff there exists two subsequences that converge to different limits. But, we will prove that any convergent subsequence of s_n has to converge to the limit of s_{2n} and s_{2n+1}, hence s_n cannot have two subsequences that converge to different limits and that will complete our proof.

Let s_{n_k} be a convergent subsequence of s_n. Then there will either exist infinitely many k for which n_k is even or infinitely many k for which n_k is odd. If there exists infinitely many k for which n_k is odd, a subsequence of s_{n_k} (the subsequence of all odd n_k) that is a subsequence of s_{2n+1} - call this subsequence $s_{n_{k_j}}$. We know that if a sequence converge, then all its subsequence will converge to the same limit. Thus,

$$\lim_{n\to\infty} s_{2n} = \lim_{n\to\infty} s_{2n+1} = \lim_{j\to\infty} s_{n_{k_j}} = \lim_{k\to\infty} s_{n_k}.$$

Similarly, if there exists infinitely many k for which n_k is even, a subsequence of s_{n_k} (the subsequence of all even n_k) that is a subsequence of s_{2n} - call this subsequence $s_{n_{k_j}}$. We know that if a sequence converge, then all its subsequence will converge to the same limit. Thus,

$$\lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} s_{2n} = \lim_{j\to\infty} s_{n_{k_j}} = \lim_{k\to\infty} s_{n_k}.$$

Now that we have some examples of convergent series, we can construct more by using the comparison theorem or by taking sum or product of known series.

THEOREM 3.29. *Given two series of real numbers* $A = \sum a_n$ *and* $B = \sum b_n$ *that converge, the sum* $A + B$ *also converge.*

Proof. Let s_n and t_n be the partial sums of A and B respectively. Further, let S_n be the partial sum of $A+B$. Then notice that $S_n = s_n + t_n$. Thus, by properties of limits, we can conclude that $\lim S_n$ exists and $\lim S_n = \lim s_n + \lim t_n$. ☐

THEOREM 3.30. *Given two series of real numbers* $A = \sum a_n$ *and* $B = \sum b_n$ *that converge, the product* AB *defined as*

$$AB = \sum_{i=0}^{\infty} \left(\sum_{k=0}^{i} a_k b_{i-k} \right)$$

also converge.

Proof. Let s_n and t_n be the partial sums of A and B respectively. Further, let P_n be the partial sum of AB. Note that

$$s_n t_n = \left(\sum_{i=0}^{n} a_n \right) \left(\sum_{i=0}^{n} b_n \right) = \sum_{i=0}^{n} \left(\sum_{k=0}^{i} a_k b_{i-k} \right) = P_n.$$

Thus, by properties of limits, we can conclude that $\lim P_n$ exists and $\lim P_n = (\lim s_n)(\lim t_n)$. ☐

EXERCISE 3.31. Show that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converge.

Although $\sum \frac{(-1)^n}{n}$ and $\sum \frac{(-1)^n}{n^2}$ both converge, we know that $\sum \frac{1}{n}$ diverge while $\sum \frac{1}{n^2}$ converge. So, in some sense, the convergence of $\frac{1}{n^2}$ is stronger. This stronger form of convergence is the theme of our next lecture.

3.3. Convergence vs Absolute convergence

As mentioned in last class, we will discuss a stronger form of convergence called absolute convergence today.

Definition 3.32 (Absolutely convergent series). Given a series of real numbers $\sum_{n=0}^{\infty} x_n$, we say the series converge absolutely, if the series $\sum_{n=0}^{\infty} |x_n|$ converges.

We claimed that this is a stronger version of convergence, so it is natural to ask if every series that converges absolutely necessarily converge. And the answer is of course, yes.

Theorem 3.33. *If the series $\sum x_n$ converge absolutely, then $\sum x_n$ converges.*

Proof. Let s_n be the sequence of partial sums of $\sum x_n$ and t_n be the sequence of partial sums of $\sum |x_n|$. As t_n converges, we know t_n is cauchy. We will prove s_n converges by proving s_n is cauchy. Assume $n > m$, then note that,

$$|s_n - s_m| = |x_{m+1} + x_{m+2} + \dots x_n| \leq |x_{m+1}| + \dots |x_n| = t_n - t_m = |t_n - t_m|.$$

Thus, if we choose N large enough such that $|t_n - t_m| < \varepsilon$ for all $n, m > N$, then $|s_n - s_m| < \varepsilon$ for all $n, m > N$. Thus, s_n is cauchy. □

As illustrated by the alternating harmonic series, there exists convergent series that are not necessarily absolutely convergent.

Definition 3.34 (Conditionally convergent series). If a series $\sum a_n$ converges but does not converge absolutely, then we say the series is conditionally convergent.

Exercise 3.35. Show that if $x_n \geq 0$, then $\sum x_n$ is convergent iff $\sum x_n$ is absolutely convergent.

Exercise 3.36. Show that if $x_n \leq 0$, then $\sum x_n$ is convergent iff $\sum x_n$ is absolutely convergent.

Exercise 3.37. Show that if $A = \sum a_n$ and $B = \sum b_n$ converge absolutely, then so does $A + B = \sum (a_n + b_n)$.

in the previous lecture we saw how to construct a new series from an existing series and this was done by grouping. Now we will see another process this is called as rearrangement.

Definition 3.38 (Rearrangement of a series). A series $\sum y_n$ is called a rearrangement of a series $\sum x_n$ if there exists a bijection $f : \mathbb{N} \to \mathbb{N}$ such that $y_k = x_{f(k)}$ for all k.

Theorem 3.39. *Let $\sum x_n$ be an absolutely convergent series, then every rearrangement $\sum y_n$ of $\sum x_n$ converge to the same value.*

Proof. Let s_n be the partial sums of $\sum x_n$, let t_n be the partial sums of $\sum y_n$, and $s = \sum x_n$. Then given any $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$ such that $|s_n - s| < \varepsilon$ for all $n \geq N$.

Notice that if $j = f^{-1}(i)$, then $y_j = x_i$. Thus, to ensure that $\{x_1, x_2, \dots, x_{N(\varepsilon)}\} \subset \{y_1, y_2, \dots, y_{M(\varepsilon)}\}$ define $M(\varepsilon) = \max\{f^{-1}(i) | 0 \leq i \leq N(\varepsilon)\}$. Thus, if $m > M(\varepsilon)$ and $n > N(\varepsilon)$, then

$$|t_m - s_n| \leq \left| \sum_{k=N(\varepsilon)+1}^{\infty} x_n \right| = |s - s_n| < \varepsilon.$$

Further,

$$|t_m - x| \leq |t_m - s_n| + |s_n - s| < 2\varepsilon.$$

As ε was arbitrary, t_m converge to s, that is, has the same limit as s_n. □

So, we proved that if a series is is absolutely convergent then any rearrangement converges and has the same value. Now, the question is, what happens if the series was conditionally convergent? In this case we will show that we can find a rearrangement such that it will converge to any value you want more precisely if you give me a value I will construct a rearrangement such that the rearranged sequence will converge to the value you had given.

Theorem 3.40. *If series $\sum x_n$ is conditionally convergent, then given any real number s we can find a rearrangement $\sum y_n$ of $\sum x_n$ such that $\sum y_n = s$.*

Let me give a brief sketch of the proof before actually jumping into the proof of the theorem. First we form two new series - we collect all the positive terms x_n and all the negative terms x_n. We will then show that the series of positive terms and series of negative terms both will have to diverge. Given this fact,

1. Choose just enough positive terms (in order - and not chosen earlier) from the positive series such that the sum of these numbers and what we have already (in the beginning we have 0) is greater than the chosen value.
2. Choose just enough number of terms from the negative series (in order - and not chosen earlier) such that the sum of these numbers and what we have already is greater than the chosen value.

Repeat the above two steps one-after-the-other again and again each time adding just enough numbers from the positive series to make it greater than the chosen value and then choosing just enough numbers from the negative series to make it slightly less than the chosen value. So at each stage we the sum overshoots and undershoots the chosen value. But, as x_n converges to 0, the amount by which it under shoots or overshoots keeps decreasing. Hence it will eventually converge it to the chosen mark this is the idea of the proof.

Lemma 3.41. *Let x_n be a conditionally convergent series. Define two sequences*

$$x_n^+ = \begin{cases} x_n & \text{if } x_n > 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$x_n^- = \begin{cases} x_n & \text{if } x_n < 0 \\ 0 & \text{otherwise} \end{cases}$$

and notice that $x_n = x_n^+ + x_n^-$. Then, $\sum x_n^+$ and $\sum x_n^-$ both diverge.

Proof. We will prove this claim by a proof by exhaustion. That is we consider all 4 cases and show that the only possibility is that both x_n^+ and x_n^- are divergent.

Case 1 - x_n^+ and x_n^- converge: Then, by Exercise 3.35 and Exercise 3.36, we can conclude that both x_n^+ and x_n^- converge absolutely. Thus, by Exercise 3.37 $\sum x_n$ converges absolutely, which is a contradiction.

Case 2 - x_n^+ is divergent, but x_n^- is convergent: Let s_n^+ be the sequence of partial sums of $\sum x_n$ and s_n^- be the sequence of partial sums for $\sum x_n^-$. As $\sum x_n^-$ is convergent, s_n^- is bounded below by some value, say $-M$, that is $s_n^- \geq -M$. On the other hand, as s_n^+ is a sequence of non-negative terms, it is divergent iff it is not bounded. Thus, given any number A, there exists a number $N(A)$ such that $s_n^+ \geq A + M$ for all $n \geq N(A)$. Thus, $s_n = s_n^+ + s_n^- \geq A + M + (-M) = A$ for all $n > N(A)$. As A was arbitrary, this implies that s_n diverges (as it is not bounded), which is a contradiction.

Case 3 - x_n^+ is convergent, but x_n^- is divergent: The proof that this case is also not possible is analogous to the proof in Case 2, hence left as an exercise.

Case 4 - x_n^+ and x_n^- are divergent: As this is the only case left, we have proved our claim. ☐

Proof of Theorem 3.40. Choose a p such that

$$\sum_{n=0}^{p-1} x_n^+ \leq s < \sum_{n=0}^{p} x_n.$$

So, it is the first p for which the sum becomes greater than s. This already tells us how to defined the function for the first p numbers. More precisely, $f(0) = \min\{n \mid x_n \neq 0 \text{ and } 0 \leq x_n \leq p\}$ (basically, we are discarding the zeroes). Similarly, $f(1) = \min\{n \mid x_n \neq 0 \text{ and } f(0) < x_n \leq p\}$. More generally, if $f(k)$ is defined, we can define $f(k+1) = \min\{n \mid x_n \neq 0 \text{ and } f(k) < x_n \leq p\}$. Of course this is not an infinite process as we can find atmost the values of the first $p+1$ natural numbers. But, it might end before $p+1$ as some of the numbers x_n^+ where $0 \leq n \leq p$ may be zero. But, we still have the function defined till some stage. Let us say, we have $f(0), \ldots, f(m_1)$ has been defined. Now we add just enough terms from the negative sequence to make the sum less that s. More presisely, we choose a q such that

$$\sum_{k=0}^{p} x_n^+ + \sum_{k=0}^{q} x_n^- < s \leq \sum_{k=0}^{p} x_n^+ + \sum_{k=0}^{q-1} x_n^-.$$

Now, define $f(m_1+1) = \min\{n \mid x_n^- \neq 0 \text{ and } 0 \leq n \leq q$. Similarly, $f(m_1+2) = \min\{n \mid x_n^- \neq 0 \text{ and } f(m_1+1) < n \leq q\}$. More generally, if we know $f(k)$, then we can define $f(k+1) = \min\{n \mid x_n^- \neq 0 \text{ and } f(m_1+k) < n \leq q\}$. In this manner, we would be able to define $f(m_1+1), \ldots, f(m_2)$ for some number m_2. And we keep on doing this process - we overshoot and undershoot. And we can step-by-step define the function $f : \mathbb{N} \to \mathbb{N}$.

As already mentioned, as x_n converges to 0, the amount by which it under shoots or overshoots keeps decreasing. Hence it will eventually converge it to the chosen mark ☐

Now that we have understood the necessary concepts in series of real numbers, we can quickly define analogous concepts for series of complex numbers. As the proofs are very similar, they are left as exercises.

Definition 3.42. Given a series of complex number $\sum_{n=0}^{\infty} z_n$, we say the series converge, if the sequence of partial sums $s_k = \sum_{n=0}^{k} z_n$ converge as k tends to infinity. And the limit is called the sum of the series.

Exercise 3.43. Let $a_n \in \mathbb{R}$ and $b_n \in \mathbb{R}$. Show that $\sum(a_n + \imath b_n)$ converge iff $\sum a_n$ and $\sum b_n$ converge.

Thus, given a complex number z, the series $\sum a_n z^n$ is a series of complex numbers and thus we can test its convergence. In the case of sequence of real numbers, the easiest way to prove convergence is by comparison with one of the examples we have discussed. However, as $\mathbb{C}$ does not have an order, we cannot compare two series. One way to tackle this problem is to use the observation in the above exercise. But, the real and imaginary parts of z^n are not that easy to find, and thus the method is not as useful. Yet another way to tackle the problem would be to notice that the absolute value of a complex number is a real number. This, in addition with the observation that absolutely convergence is far more well-behaved suggests that we focus on absolute convergence. We will later see an even more pressing reason to focus on absolute convergence. Then comparison with the examples we have studied will work well.

Definition 3.44. We say the series $\sum z_n$ converge absolutely, if the series $\sum |z_n|$ converge.

3.4. Radius of convergence

In the last section, we discussed two forms of convergence: conditional and absolute. We further discussed that conditional convergence does not behave well under rearrangement and thus is trickier. In addition, the absolute value of a complex number is a real number, so we can use techniques from real analysis to study the absolute convergence of a series of complex numbers. For instance, we have the following theorem.

Theorem 3.45 (Comparison Theorem). *Assume $\sum_{n=1}^{\infty} x_n$ is a convergent series of positive real numbers and $|z_n| \leq x_n$ for all n. Then, $\sum_{n=1}^{\infty} z_n$ converges absolutely.*

Proof. As the series $\sum_{n=1}^{\infty} x_n$ converge, its value is some number, say A. We know that the series of partial sums $t_m = \sum_{n=1}^{m} |z_n|$ is a monotonically increasing sequence. Thus, to prove it is convergent, it is enough to show it is bounded above. And, note that,

$$t_m = \sum_{n=1}^{m} |z_n| \leq \sum_{n=1}^{m} x_n \leq \sum_{n=1}^{\infty} x_n = A.$$

Thus, the sequence of partial sums t_m is bounded above by A and hence convergent. ☐

Recall that all this discussion on the convergence of a series of real/complex numbers was motivated by the desire to treat formal power series as functions. Further recall that given a formal power series $f(z) = \sum_{n=1}^{\infty} a_n z^n$, we wanted to think of the series as a limit of the sequence of functions $f_m(z) = \sum_{n=1}^{m} a_n z^n$. And the "weakest" limit of a sequence of functions is the pointwise-limit. Thus, we would like to define the function f at a point z_0 if the sequence $f_m(z_0)$ converges. Or on the other hand, the series $\sum_{n=1}^{\infty} a_n (z_0)^n$ converges.

Theorem 3.46. *If a power series $\sum a_n z^n$ converges for some z_0, then it converges absolutely for all z such that $|z| < |z_0|$.*

Proof. As the series $\sum a_n (z_0)^n$ converges, its n-th term $a_n (z_0)^n$ converges to 0 and hence $|a_n (z_0)^n|$ converge to 0. Thus, we can say that there exist some A such that $|a_n (z_0)^n| < A$ for all n. Now,

$$|a_n z^n| = |a_n||z|^n \leq \left(\frac{A}{|z_0|^n}\right) |z|^n = A \left|\frac{z_0}{z}\right|^n.$$

Let $r = \left|\frac{z_0}{z}\right| < 1$. Thus, the convergence of the geometric series (Lecture 11) and the comparison theorem implies that the series $\sum a_n z^n$ converges absolutely if $|z| < |z_0|$. ☐

Theorem 3.47. *Assume the power series $\sum a_n z^n$ does not converge absolutely for all z, that is, there exists some z_0 such that $\sum a_n (z_0)^n$ does not converge absolutely. Then, there exists a positive real number r such that $\sum a_n z^n$ converge absolutely if $|z| < r$ and does not converge absolutely if $|z| > r$.*

Proof. Define $S := \{x \in \mathbb{R}_{\geq 0} : \sum a_n x^n \text{ converge}\}$. Notice that the set S is non-empty as 0 always belongs to S. On the other hand, S is bounded above by $|z_0|$ (by Comparison theorem). Thus, S has a least upper bound, let this least upper bound be r. As r is the supremum of S, if you take $t = |z| > r$, then $\sum a_n t^n$ does not converge. That is, the series $\sum a_n z^n$ does not converge absolutely. On the other hand, if $t = |z| < r$, then t is not an upper bound for S. That means there exists an $s \in S$ such that $s > t$. As $s \in S$, $\sum a_n s^n$ converge and hence by the comparison theorem $\sum a_n t^n$ converge. That is, the series $\sum a_n z^n$ converge absolutely. ☐

Definition 3.48 (Radius of convergence). Let $\sum a_n z^n$ be a power series. If the series converges absolutely for all z, then we will say the the radius of convergence is infinity. Else, the unique r given by Theorem 3.47 is called the radius of convergence.

In fact, Theorem 3.46 helps us make the implication of Theorem 3.47 a bit stronger.

Theorem 3.49. *Let* $\sum a_n z^n$ *be a power series and* r *its radius of convergence. If* $|z_0| > r$*, then* $\sum a_n (z_0)^n$ *does not converge (not even conditionally).*

Proof. Let us use proof by contradiction. Assume that $\sum a_n (z_0)^n$ converge. Then, by Theorem 3.46, $\sum a_n z^n$ will converge absolutely if $|z| < |z_0|$. As $|z_0| > r$, there exists z_1 such that $r < |z_1| < |z_0|$. Our assertion that $\sum a_n z^n$ will converge absolutely if $|z| < |z_0|$ implies that $\sum a_n (z_1)^n$ converge absolutely. As $|z_1| > r$, this contradicts the definition of radius of convergence. ☐

Remark 3.50. Thus, given a power series $\sum a_n z^n$ with radius of convergence r, $\sum a_n (z_0)^n$ can converge conditionally only if $|z_0| = r$. Hence, it makes even more sense to focus only on absolute convergence.

Example 3.51. Consider the power series $\sum z^n$. We know that the series does not converge if $z = 1$. Thus, the radius of convergence is less than or equal to 1. On the other hand, $|z| < 1$, then comparison with the geometric series tells us that $\sum a_n z^n$ is absolutely convergent. Hence, $[0, 1) \subset S = \{x \in \mathbb{R}_{\geq 0} \ : \ \sum a_n x^n \text{ converges}\}$. Therefore, the radius of convergence is greater than or equal to 1. Hence, the radius of convergence should be equal to 1.

Example 3.52. Consider the power series $\sum n^n z^n$. Notice that if $|z| \neq 0$, then for large enough n, $n|z| > 1$ and hence the sequence $(n|z|)^n$ does not converge to 0. Thus, the series $\sum n^n z^n$ does not converge absolutely if $|z| > 0$. Therefore the radius of convergence is equal to 0.

So, we would like to find the radius of convergence of any power series. There is a nice expression for the radius of convergence of a power series, but it uses the notion of lim sup.

Definition 3.53. Let $(x_n)_{n=1}^{\infty}$ be a bounded sequence of real numbers. As the sequence is bounded above, Bolzano-Weierstrass theorem says that there exist a convergent subsequence. Thus, the collection of sub-sequential limits S is non-empty and bounded above by every upper bound of the sequence. Thus, S has a supremum. Define,

$$\limsup_{n \to \infty} x_n = \sup(S).$$

Exercise 3.54. Compute the lim sup of the following sequences.

(1) $x_n = \frac{1}{n}$
(2) $x_n = (-1)^n$

Exercise 3.55. Consider a bounded sequence x_n. Let $s = \sup\{x_n : n \in \mathbb{N}\}$. Show that if $s \notin \{x_n : n \in \mathbb{N}\}$, then $s = \limsup x_n$.

First, we would like another characterisation of lim sup, which would prove more helpful.

Theorem 3.56. *Let* x_n *be a bounded sequence. Then* $\lambda = \limsup(x_n)$ *iff given* $\varepsilon > 0$*, there exists only finitely many* n *such that* $x_n \geq \lambda + \varepsilon$ *and there exists infinitely many* n *such that* $x_n \geq \lambda - \varepsilon$

Proof. First we will assume $\lambda = \limsup(x_n)$ and prove that given $\varepsilon > 0$, there exists only finitely many n such that $x_n \geq \lambda + \varepsilon$ and there exists infinitely many n such that $x_n \geq \lambda - \varepsilon$.

If there exists infinitely many n such that $x_n \geq \lambda + \varepsilon$, then these elements form a subsequence x_{n_k} such that $x_{n_k} \geq \lambda + \varepsilon$. However, as x_n is bounded, so is x_{n_k}. Thus, by Bolzano-Wiestrass theorem, some subsequence of x_{n_k} will converge. As we could have started with this subsequence, we will continue to denote the subsequence as x_{n_k}. As $x_{n_k} \geq \lambda + \varepsilon$, $\lim x_{n_k} \geq \lambda + \varepsilon$. That is, there is a subsequential limit greater than $\lambda + \varepsilon$ which contradicts the assumption that $\lambda = \limsup(x_n)$.

On the other hand, as $\lambda = \limsup(x_n)$, given $\varepsilon > 0$, $\lambda - \frac{\varepsilon}{2}$ is not an upper bound for subsequential limits. Thus, there exists some subsequence x_{n_k} which converge to a value x greater than $\lambda - \frac{\varepsilon}{2}$. Further, as x_{n_k} converges to x, there exists some N such that

$$\lambda - \varepsilon = \left(\lambda - \frac{\varepsilon}{2}\right) - \frac{\varepsilon}{2} < x - \frac{\varepsilon}{2} < x_{n_k} < x + \frac{\varepsilon}{2}$$

for all $k \geq N$. Thus, we have infinitely many n (x_{n_k} for $k \geq N$) such that $x_n \geq \lambda - \varepsilon$.

Now, we will assume λ is a number such that there exists only finitely many n such that $x_n \geq \lambda + \varepsilon$ and there exists infinitely many n such that $x_n \geq \lambda - \varepsilon$ and prove $\lambda = \limsup(x_n)$.

Fix an $\varepsilon > 0$ arbitrarily. As there are only finitely many n such that $x_n \geq \lambda + \varepsilon$, all subsequential limits have to be less than or equal to $\lambda + \varepsilon$. Hence, $\limsup(x_n) \leq \lambda + \varepsilon$. As $\varepsilon > 0$ was arbitrary, this implies $\limsup(x_n) \leq \lambda$.

On the other hand, as there are infinitely many n for which $x_n \geq \lambda - \varepsilon$, Bolzano-Wiestrass theorem implies (as discussed earlier) that there is a subsequential limit greater than or equal to $\lambda - \varepsilon$. Hence, $\limsup(x_n) \geq \lambda - \varepsilon$. As $\varepsilon > 0$ was arbitrary, this implies $\limsup(x_n) \geq \lambda$.

Combining the two inequalities we have $\lambda = \limsup(x_n)$. □

THEOREM 3.57. *Let $\sum a_n z^n$ be a power series and let r be its radius of convergence. Then*

$$\frac{1}{r} = \limsup |a_n|^{\frac{1}{n}}.$$

Proof. Let $t = \limsup |a_n|^{\frac{1}{n}}$ and $t \neq 0, \infty$. Given $\varepsilon > 0$, there exists only finitely many n such that $|a_n|^{\frac{1}{n}} \geq t + \varepsilon$. Thus, for all but finitely many n, we have $|a_n| \leq (t+\varepsilon)^n$. Let k be such that $|a_n| \leq (t+\varepsilon)^n$. Thus, we have

$$\begin{aligned}
\sum |a_n z^n| &\leq \sum_{n=0}^{k} |a_n z^n| + \sum_{n=k}^{\infty} |a_n z^n| \\
&\leq \sum_{n=0}^{k} |a_n||z|^n + \sum_{n=k}^{\infty} ((t+\varepsilon)|z|)^n \\
&= \sum_{n=0}^{k} ((t+\varepsilon)|z|)^n + \sum_{n=0}^{k} \left(|a_n| - (t+\varepsilon)^n\right)|z|^n + \sum_{n=k}^{\infty} ((t+\varepsilon)|z|)^n \\
&= A + \sum_{n=0}^{\infty} (|z|(t+\varepsilon))^n
\end{aligned}$$

where

$$A = \sum_{n=0}^{k} \left(|a_n| - (t+\varepsilon)^n\right)|z|^n.$$

As the series $\sum (|z|(t+\varepsilon))^n$ converge if $|z|(t+\varepsilon) < 1$, or in other words, $|z| < \frac{1}{t+\varepsilon}$. Thus, $r \geq \frac{1}{t+\varepsilon}$ for all $\varepsilon > 0$. Thus, $r \geq \frac{1}{t}$.

Conversely, given ε, there exists infiniely many n such that $|a_n|^{\frac{1}{n}} \geq t - \varepsilon$. Hence, the series does not converge if $|z| = t - \varepsilon$. Therefore, $r \leq \frac{1}{t-\varepsilon}$ for all $\varepsilon > 0$. Thus, $r \leq \frac{1}{t}$.

Combining the two inequalities, we have the required result. □

THEOREM 3.58. *Let a_n be a sequence of positive numbers, and assume that*

$$\lim \frac{a_{n+1}}{a_n} = A \geq 0. \text{ Then } \lim (a_n)^{\frac{1}{n}} = A.$$

Proof. Given $\varepsilon > 0$, there exists an n_0 such that

$$A - \varepsilon \leq \frac{a_{n+1}}{a_n} \leq A + \varepsilon$$

if $n \geq n_0$. Thus,

$$A - \varepsilon \leq \frac{a_{n_0+1}}{a_{n_0}} \leq A + \varepsilon,$$

that is $a_{n_0}(A-\varepsilon) \leq a_{n_0+1} \leq a_{n_0}(A+\varepsilon)$. Similarly,

$$A - \varepsilon \leq \frac{a_{n_0+2}}{a_{n_0+1}} \leq A + \varepsilon,$$

that is $a_{n_0+1}(A-\varepsilon) \leq a_{n_0+2} \leq a_{n_0+1}(A+\varepsilon)$. But, $a_{n_0+1} \geq a_{n_0}(A-\varepsilon)$ and $a_{n_0+1} \leq a_{n_0}(A+\varepsilon)$. Thus, $a_{n_0}(A-\varepsilon)^2 \leq a_{n_0+2} \leq a_{n_0}(A+\varepsilon)^2$. More generally, using induction, we can prove $a_{n_0}(A-\varepsilon)^k \leq a_{n_0+k} \leq a_{n_0}(A+\varepsilon)^k$. Thus, $a_{n_0}(A-\varepsilon)^{-n_0}(A-\varepsilon)^n \leq a_n \leq a_{n_0}(A+\varepsilon)^{-n_0}(A+\varepsilon)^n$. Define $C_1(\varepsilon) = a_{n_0}(A-\varepsilon)^{-n_0}$ and $C_2(\varepsilon) = a_{n_0}(A+\varepsilon)^{-n_0}$. Then, we have,

$$C_1(\varepsilon)(A-\varepsilon)^n \leq a_n \leq C_2(\varepsilon)(A+\varepsilon)^n.$$

Case 1: $A > 0$
Take $\varepsilon < A$. Then, $(A-\varepsilon)$ is positive and hence we can take n-roots of $C_i(\varepsilon)$ and $A-\varepsilon$. As $C_1(\varepsilon)^{\frac{1}{n}}$ and $C_2(\varepsilon)^{\frac{1}{n}}$ both tend to 1 as n tends to infinity, we get,

$$A - \varepsilon \leq (a_n)^{\frac{1}{n}} \leq A + \varepsilon$$

for all $\varepsilon > 0$.

Case 2: $A = 0$
As a_n is positive and $A - \varepsilon < 0$, we can improve our inequalities to get

$$0 \leq a_n \leq C_2(\varepsilon)(A+\varepsilon)^n.$$

As $C_2(\varepsilon)^{\frac{1}{n}}$ tend to 1 as n tends to infinity, we get,

$$0 \leq (a_n)^{\frac{1}{n}} \leq A + \varepsilon$$

for all $\varepsilon > 0$.

Thus, in either case we have proved that $(a_n)^{\frac{1}{n}}$ converges to A. □

We can use this observation to construct a series that converge absolutely for all $z \in \mathbb{C}$. Notice that if

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 0$$

then the power series will converge for all $z \in \mathbb{C}$. As we are choosing the series, we can ofcourse choose $a_n > 0$ for all n and $a_0 = 1$. And, the simplest example of a sequence that converge to 0 is the obviously the sequence $\frac{1}{n}$. Thus, if we want $\lim (a_{n+1}/a_n) = 0$, a natural option would be to set $a_{n+1}/a_n = 1/n$. However, as the indices for power series start with 0 instead of 1, this leads to the problem of division by 0. Thus, we make a small modification and set $a_{n+1}/a_n = 1/(n+1)$. That is, $a_{n+1} = a_n/(n+1)$. Thus,

$$a_1 = \frac{a_0}{1} = 1$$

$$a_2 = \frac{a_1}{2} = \frac{1}{2}$$

$$a_3 = \frac{a_2}{3} = \frac{1}{3!}$$

$$a_4 = \frac{a_3}{4} = \frac{1}{4!}$$

and more generally, using induction, you may prove that

$$a_n = \frac{1}{n!}$$

This tells us that the power series

$$\sum_{n=0}^{\infty} \frac{1}{n!} z^n$$

converge for all z. But, recall that this is the familiar exponential series. Of course we are interested in the exponential function because of its practical use (that it is ubiquitous in nature). The fact that a purely theoretical speculation gives the power series representation of THAT function seems magical. This observation was made by my friend G.P Balakumar and I was completely mesmerised when I first heard it.

The explicit formula for computing the radius of convergence also allows us to infer convergence of sum, product and quotient of two power series. We will end this lecture discussing these properties.

LEMMA 3.59. *Suppose $\sum a_n z^n$ has a radius of convergence greater than 0. Then given $0 < s < r$, there exists $C > 0$ such that $|a_n| s^n \le C$ for all n.*

Proof. As $s < r$, we know $\sum |a_n| s^n$ converge absolutley. Thus, $\lim |a_n| s^n$ converge to 0 and hence is bounded. □

THEOREM 3.60. *If $f = \sum a_n z^n$ and $g = \sum b_n z^n$ are power series that converge absolutely on $D(0,r)$, the $f+g$ converge absolutely on $D(0,r)$.*

Proof. Applying Lemma 3.59 to the two series, we get C_1 such that $|a_n| s^n < C_1$ and C_2 such that $|b_n| s^n < C_2$. Let C be the maximum of C_1 and C_2. Then $|a_n| < \frac{C}{s^n}$ and $|b_n| < \frac{C}{s^n}$. Thus, $|a_n + b_n| \le |a_n| + |b_n| < \frac{2C}{s^n}$. As, $(2C)^{\frac{1}{n}}$ converge to 1 as n tend to infinity, we get $\limsup |a_n + b_n|^{\frac{1}{n}} < \frac{1}{s}$. Hence the radius of convergence of $f+g$ is greater than or equal to s. As $s < r$ was arbitrary, radius of convergence of $f+g$ is greater than or equal to r. □

THEOREM 3.61. *If $f = \sum a_n z^n$ and $g = b_n z^n$ are power series that converge absolutely on $D(0,r)$, the $fg = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} a_k b_{n-k}\right) z^n = \sum c_n z^n$ converge absolutely on $D(0,r)$.*

Proof. Applying Lemma 3.59 to the two series, we get C_1 such that $|a_n| s^n < C_1$ and C_2 such that $|b_n| s^n < C_2$. Let C be the maximum of C_1 and C_2. Then $|a_n| < \frac{C}{s^n}$ and $|b_n| < \frac{C}{s^n}$. Thus,

$$|c_n| \le \sum_{k=0}^{n} |a_k||b_{n-k}| \le \sum_{k=0}^{\infty} \frac{C}{s^k} \frac{C}{s^{n-k}} = \frac{C^2}{s^n}.$$

As, $(C^2)^{\frac{1}{n}}$ converge to 1 as n tend to infinity, we get $\limsup |c_n|^{\frac{1}{n}} < \frac{1}{s}$. Hence the radius of convergence of fg is greater than or equal to s. As $s < r$ was arbitrary, radius of convergence of fg is greater than or equal to r. $\square$

THEOREM 3.62. *Suppose $f = \sum a_n z^n$ has non-zero constant term and non-zero radius of convergence, then $\frac{1}{f}$ has non-zero radius of convergence.*

Proof. If $a_0 \neq 0$, then $g = a_0^{-1} f$ has a non-zero radius of convergence and the constant term is equal to 1. Notice that $\frac{1}{f} = a_0 \frac{1}{g}$. Thus, it is enough to prove that $\frac{1}{g}$ has a non-zero raidus of convergence. In other words, we may assume that the constant term is 1.

Thus, $f = 1 + a_1 z + a_2 z^2 + \cdots = 1 - h(z)$. Therefore $\frac{1}{f(z)} = 1 + h(z) + h(z)^2 + \ldots$. By Lemma 3.59 we know that $|a_n| \leq \frac{C}{s^n}$. That is $|a_n| < \left(\frac{C^{\frac{1}{n}}}{s}\right)^n$. Denote $A = \frac{C^{\frac{1}{n}}}{s}$ and we have $|a_n| < A^n$. So, if $|z| < \frac{1}{2}$, then

$$|h(z)| \leq \left|\sum_{n=1}^{\infty} A^n z^n\right| = \left|\frac{Az}{1 - Az}\right|.$$

Thus, the power series $1 + h(z) + h(z)^2 + \ldots$ converge if

$$\left|\frac{Az}{1 - Az}\right| < 1.$$

That is $|Az| < |1 - Az|$. We know $|1 - Az| \geq |1 - |Az||$. Thus, it is enough to ensure $|Az| < |1 - |Az||$. In other words, $1 - |Az| < -|Az|$ or $1 - |Az| > |Az|$. As the first inequality is impossible, it is enough to check for what values of z is $1 - |Az| > |Az|$. And this happens precisely when $|z| < 1/(2A)$ $\square$

3.5. Point-wise convergence

We intuitively feel that given a formal power series $f = \sum a_n z^n$, the sequence of polynomials $f_k = \sum_{n=0}^{k} a_n z^n$ converge to the power series f. To make this precise, we need a formal definition of convergence of a sequence of functions. Convergence of a sequence of functions can be defined in multiple ways and we will start with the simplest definition.

DEFINITION 3.63. We say $f_n : \mathbb{R} \to \mathbb{R}$ **converges pointwise** to $f : \mathbb{R} \to \mathbb{R}$ if $f_n(x)$ converges to $f(x)$ for all $x \in \mathbb{C}$.

EXAMPLE 3.64. Consider the function $f_n : \mathbb{R} \to \mathbb{R}$ defined as $f_n(x) = \left(1 - \frac{1}{n}\right) x$. It is clear that for each x, the sequence $\left(1 - \frac{1}{n}\right) x$ converge to x.

More generally,

EXAMPLE 3.65. A sequence of polynomials $a_{k,n} z^k + a_{k-1,n} z^{k-1} + \ldots a_{0,n}$ will converge pointwise to the polynomial $a_k z^k + a_{k-1} z^k - 1 + \ldots a_0$, if $a_{i,n}$ converge to a_i for $0 \leq i \leq k$.

EXAMPLE 3.66. Let $f_n : \mathbb{R} \to \mathbb{R}$ be the function defined as

$$f_n(x) = \begin{cases} -1 & \text{if } x \leq \frac{-1}{n} \\ nx & \text{if } |x| < \frac{1}{n} \\ 1 & \text{if } x \geq \frac{+1}{n} \end{cases}.$$

Given any $x > 0$, you can find an N large enough such that $x \geq \frac{1}{N}$. Thus, $f_n(x) = 1$ for all $n \geq N$. Thus, we know that $f_n(x)$ converge to 1. Similarly, if $f_n(x) < 0$, then $f_n(x)$ converge to -1. On the other hand, $f_n(0) = 0$ for all n. Thus, $f_n(0)$ converge to the function 0. Thus, the sequence of functions f_n converge to the function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}.$$

Notice that each f_n is continuous, but the limit f is not continuous. We opt for stronger notions of convergence to ensure that the limit inherits certain nice properties.

Example 3.67. Let $f_n : \mathbb{R} \to \mathbb{R}$ be the function defined as $f_n(x) = \frac{1}{n}\chi_{(0,n)}$. If $x \leq 0$, the sequence $f_n(x)$ is the constant sequence containing just 0. On the other hand, if $x > 0$, then eventually (once x becomes less than n) it becomes the constant sequence 0. Thus, the whole sequence converges to 0.

Notice something interesting - $\int f_n = 1$ for all n, but $\int f = 0$. Thus, this is an example of a sequence of functions such that $f_n \to f$ but $\int f_n \not\to \int f$. Once again, if f_n converges to f, we would like $\int f_n$ to converge to $\int f$. After all, the Riemann integral of a function f is obtained as a the limit of the integrals of a sequence of step functions f_n that converge to f.

Example 3.68. Let $f_n : \mathbb{R} \to \mathbb{R}$ be the function defined as $f_n(x) = \chi_{(n,n+1)}$. Eventually (once x becomes less than n) the sequence $f_n(x)$ becomes the constant sequence 0. Thus, the whole sequence converges to 0.

Example 3.69. Let $f_n : \mathbb{R} \to \mathbb{R}$ be the function defined as $f_n(x) = n\chi_{[0,\frac{1}{n}]}$. If $x < 0$, the sequence $f_n(x)$ is the constant sequence containing just 0. On the other hand, if $x > 0$, then eventually (once x becomes greater than $\frac{1}{n}$) it becomes the constant sequence 0. Thus, the whole sequence converges to 0. However, when $x = 0$, the sequence $f_n(x) = n$ and therefore does not converge. Therefore, the sequence $f_n(x)$ does not converge pointwise to any function f.

Example 3.70. Let $f_n : \mathbb{R} \to \mathbb{R}$ be the function defined as

$$f_n(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x^n & \text{if } x \in (0,1) \\ 1 & \text{if } x \geq 1 \end{cases}.$$

If $x \notin (0,1)$, then the sequence $f_n(x)$ is a constant sequence. If $x \in (0,1)$, then the sequence is not constant, but we know it converges to 0. Thus, the limit is the function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} 0 & \text{if } x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}.$$

Example 3.71. Let $f_n : \mathbb{R} \to \mathbb{R}$ be the function defined as

$$f_n(x) = \begin{cases} \frac{1}{nx} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

and $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = 0$. If $x \leq 0$, the sequence $f_n(x)$ is the constant sequence containing just 0. On the other hand, if $x > 0$, then $\frac{1}{x}$ is a positive number c and $\frac{c}{n}$ converges to 0. Thus, f_n converges to f pointwise.

Example 3.72. Let r_i be an enumeration of rational numbers. Define

$$f_n(x) = \begin{cases} 1 & \text{if } x = r_i \text{ for some } i < n \\ 0 & \text{otherwise} \end{cases}.$$

Notice that the sequence $f_n(x)$ converge to the Dirichlet function

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}.$$

Notice that each of the functions f_n is discontinuous only at finitely many points, but the limit is discontinuous everywhere!

3.5.1. Devil's staircase. Consider a recursively defined sequence of function, $f_0(x) = x$ and

$$f_{n+1}(x) = \begin{cases} \frac{f_n(3x)}{2} & \text{if } x \in [0, \frac{1}{3}] \\ \frac{1}{2} & \text{if } x \in [\frac{1}{3}, \frac{2}{3}] \\ \frac{f_n(3x-2)+1}{2} & \text{if } x \in [\frac{2}{3}, 1] \end{cases}.$$

Let $f : [0,1] \to \mathbb{R}$ be the function defined as follows. If a belongs to the Cantor set, that is a is of the form $a = \sum_{n=1}^{\infty} \frac{2a_n}{3^n}$ for $a_n \in \{0,1\}$, then $f(a) = \sum_{n=1}^{\infty} \frac{a_n}{2^n}$. If $a = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$ does not belong the Cantor set. Define $N = \min\{n \in \mathbb{N} : a_n = 1\}$ and $f(a) = \sum_{i=1}^{N-1} \frac{a_i}{2^{i+1}} + \frac{1}{2^N}$.

Lemma 3.73. *If a belongs to the Cantor set, that is a is of the form $a = \sum_{i=1}^{\infty} \frac{2a_i}{3^i}$ for $a_i \in \{0,1\}$, then,*

$$f_n\left(\sum_{i=1}^{\infty} \frac{2a_i}{3^i}\right) = \frac{f_0\left(\sum_{i=n+1}^{\infty} \frac{2a_i}{3^{i-n}}\right)}{2^n} + \sum_{i=1}^{n} \frac{a_i}{2^i}$$

Proof. We will prove the claim using induction. First notice that, $a \in [0, \frac{1}{3}]$ iff $a_1 = 0$. Thus, if $a_1 = 0$,

$$f_m(a) = \frac{f_{m-1}\left(3 \times \sum_{i=1}^{\infty} \frac{2a_i}{3^i}\right)}{2} = \frac{f_{m-1}(2a_1 + \sum_{i=2}^{\infty} \frac{2a_i}{3^{i-1}})}{2} = \frac{f_{m-1}(\sum_{i=2}^{\infty} \frac{2a_i}{3^{i-1}})}{2}.$$

Similarly, $a \in [\frac{2}{3}, 1]$ iff $a_1 = 1$. Thus, if $a_1 = 1$,

$$f_m(a) = \frac{f_{m-1}\left(3 \times \sum_{i=1}^{\infty} \frac{2a_i}{3^i} - 2\right)}{2} + \frac{1}{2} = \frac{f_{m-1}(2a_1 - 2 + \sum_{i=2}^{\infty} \frac{2a_i}{3^{i-1}})}{2} + \frac{1}{2} = \frac{f_{m-1}(\sum_{i=2}^{\infty} \frac{2a_i}{3^{i-1}})}{2} + \frac{1}{2}.$$

Combining the two results, we have

$$f_m(a) = \frac{f_{m-1}(\sum_{i=2}^{\infty} \frac{2a_i}{3^{i-1}})}{2} + \frac{a_1}{2}.$$

Thus, by taking $m = 0$, we have proved base case of the theorem, that is when $n = 1$. We will now assume the statement is true when $n = k$ and prove the statement for $n = k + 1$.

$$\begin{aligned} f_{k+1}(a) &= \frac{f_k\left(\sum_{i=2}^{\infty} \frac{2a_i}{3^{i-1}}\right)}{2} + \frac{a_1}{2} \\ &= \frac{\frac{f_0\left(\sum_{i=k+2}^{\infty} \frac{2a_i}{3^{i-k-1}}\right)}{2^k} + \sum_{i=2}^{k+1} \frac{a_i}{2^i}}{2} + \frac{a_1}{2} \\ &= \frac{f_0\left(\sum_{i=k+2}^{\infty} \frac{2a_i}{3^{i-k-1}}\right)}{2^{k+1}} + \sum_{i=1}^{k+1} \frac{a_i}{2^i} \end{aligned}$$

□

Notice that $\frac{a_i}{2^i} \leq \frac{1}{2^{i-1}}$ and $\sum \frac{1}{2^{i-1}}$ converges to 2. Thus, $\sum \frac{a_i}{2^i}$ converges as n tends to zero. Moroever, $\frac{f_0\left(\sum_{i=n+1}^{\infty} \frac{2a_i}{3^{i-n}}\right)}{2^n}$ converges to 0 as n tends to ∞. Thus, we have

Lemma 3.74. *If a belongs to the Cantor set, that is a is of the form $a = \sum_{i=1}^{\infty} \frac{2a_i}{3^i}$ for $a_i \in \{0, 1\}$, then, $f_n\left(\sum_{i=1}^{\infty} \frac{2a_i}{3^i}\right)$ converges to $\sum_{i=1}^{\infty} \frac{a_i}{2^i}$.*

Lemma 3.75. *If $a_1 = 1$, then $a = \sum_{i=1}^{\infty} \frac{a_i}{3^i} \in [\frac{1}{3}, \frac{2}{3}]$. Thus, $f_n(a) = \frac{1}{2}$ for all n.*

Lemma 3.76. *Let $a = \sum_{i=1}^{\infty} \frac{a_i}{3^i}$ belong to the complement of the Cantor set. Define $N = \min\{i \in \mathbb{N} : a_i = 1\}$. Then, for $n > N$,*

$$f_n(a) = \sum_{i=1}^{N-1} \frac{a_i}{2^{i+1}} + \frac{1}{2^N}.$$

As $f_n(a)$ is eventually constant, it converges to the same value.

Proof. As $a_i \in \{0, 2\}$ when $i < N$, using the same arguments as before, when $m < N$

$$f_{m+k}\left(\sum_{i=1}^{\infty} \frac{a_i}{3^i}\right) = \frac{f_k\left(\sum_{i=n+1}^{\infty} \frac{a_i}{3^{i-n}}\right)}{2^m} + \sum_{i=1}^{m} \frac{a_i}{2^{i+1}}.$$

Thus, given $n > N$, take $m = N - 1$ and $k = n - N + 1$. Then, we have

$$f_n\left(\sum_{i=1}^{\infty} \frac{a_i}{3^i}\right) = \frac{f_{n-N+1}\left(\sum_{i=N}^{\infty} \frac{a_i}{3^{i-N+1}}\right)}{2^{N-1}} + \sum_{i=1}^{N-1} \frac{a_i}{2^{i+1}}.$$

But, by previous lemma $f_{n-N+1}\left(\sum_{i=N}^{\infty} \frac{a_i}{3^{i-N+1}}\right) = \frac{1}{2}$ for all $n > N$ and thus we have the result. ☐

Combining Lemma 3.74 and Lemma 3.76 we have

Theorem 3.77. *The sequence f_n converges pointwise to f.*

3.5.2. Galloping rectangles function. Let $N = \lfloor \log_2(n) \rfloor$ and $k = n \mod (2^N)$. Define

$$f_n(x) = \begin{cases} 1 & \text{if } \frac{k}{2^N} \le x \le \frac{k+1}{2^N} \\ 0 & \text{otherwise} \end{cases}$$

Remark 3.78. Recall we saw that every inner product induced a norm, but not all norms are induced by an inner product. Every norm induces a metric, but not all metrics are induced by a norm. We finally saw that every metric induces a topology. So, it is natural to ask if every topology is induced by a metric.

When talking about topology of the complex plane, I talked as if defining a notion of convergence is equivalent to defining openness. However, this is not true and I was lightly cheating you.

3.6. Uniform convergence

Definition 3.79. We say a sequence $f_n : \mathbb{R} \to \mathbb{R}$ **converges uniformly** (or L^∞ convergence) to a function $f : \mathbb{R} \to \mathbb{R}$ iff

$$\lim_{n\to\infty} \sup\{|f_n(x) - f(x)| : x \in \mathbb{R}\} = 0.$$

Theorem 3.80. *Let $\sum a_n z^n$ be a power series and r be its radius of convergence. Let $f : D(0, r) \to \mathbb{C}$ be the function $f(z) = \sum a_n z^n$ and $f_m : D(0, r) \to \mathbb{C}$ be the function $f_m(z) = \sum_{n=1}^{m} a_n z^n$. If $0 < \rho < r$ Then, the sequence of functions f_m converge unifomrly to the function f on the set $D(0, \rho)$. That is,*

$$\lim_{m\to\infty} \sup\{|f(z) - f_m(z)| \ : \ z \in D(0, \rho)\} = 0.$$

Proof. As $\rho < r$, we know $\sum a_n \rho^n$ converge. Thus, given any ε, there exists an N such that $\sum_{n=N+1}^{\infty} < \varepsilon$. Now notice that

$$|f(z) - f_m(z)| = \left| \sum_{n=m+1}^{\infty} a_n z^n \right| \leq \sum_{n=m+1}^{\infty} |a_n||z|^n \leq \sum_{n=m+1}^{\infty} |a_n|\rho^n.$$

Thus, for all $z \in D(0, \rho)$, $|f(z) - f_m(z)| < \varepsilon$ if $m > N$. that is,

$$\lim_{m\to\infty} \sup\{|f(z) - f_m(z)| \ : \ z \in D(0,\rho)\} = 0.$$

□

EXAMPLE 3.81. Consider the function $f_n : \mathbb{R} \to \mathbb{R}$ defined as $f_n(x) = \left(1 - \frac{1}{n}\right) x$. Notice that

$$\sup\{|f_n(x) - f(x)| : x \in \mathbb{R}\} = \sup \left\{ \left| \frac{x}{n} \right| \right\} = \infty.$$

Thus, this sequence does not converge uniformly. However, if $f_n : [a, b] \to \mathbb{R}$ defined as $f_n(x) = \left(1 - \frac{1}{n}\right) x$ converges uniformly as

$$\sup\{|f_n(x) - f(x)| : x \in \mathbb{R}\} = \sup \left\{ \left| \frac{x}{n} \right| \right\} = \max\{|a|, |b|\}.$$

EXAMPLE 3.82. Let $f_n : \mathbb{R} \to \mathbb{R}$ be the function defined as

$$f_n(x) = \begin{cases} -1 & \text{if } x \leq \frac{-1}{n} \\ nx & \text{if } |x| < \frac{1}{n} \\ 1 & \text{if } x \geq \frac{+1}{n} \end{cases}.$$

Then,

$$\begin{aligned} \sup\{|f_n(x) - f(x)| : x \in \mathbb{R}\} &\leq \sup \left\{ |f_n(x) - f(x)| : x \in \left[-\frac{1}{n}, \frac{1}{n}\right] \right\} \\ &= \left\{ |nx - 0| : x \in \left[-\frac{1}{n}, \frac{1}{n}\right] \right\} = 1. \end{aligned}$$

The convergence is non-uniform can also be concluded from a more general result on the continuity of the uniform limit of continuous functions.

THEOREM 3.83. *Let $A \subset \mathbb{C}$ and let $f_m : A \to \mathbb{C}$ be a sequence of continuous function that converge uniformly to the function $f : A \to \mathbb{C}$. Then, f is also continuous.*

Proof. Let $\alpha \in A$ be arbitrary. We will prove f is continuous at α. In other words, we want to prove, given any ε, there exists a δ such that $|f(z) - f(\alpha)| < \varepsilon$ if $|z - \alpha| < \delta$. Notice that

$$|f(z) - f(\alpha)| \leq |f(z) - f_m(z)| + |f_m(z) - |f_m(\alpha)| + |f_m(\alpha) - f(\alpha)|.$$

Further, $|f(z) - f_m(z)|$ and $|f_m(\alpha) - f(\alpha)|$ are both less than or equal to $\sup\{|f(z) - f_m(z)| \ : \ z \in A\}$. As $\sup\{|f(z) - f_m(z)| \ : \ z \in A\}$ converges to 0, we can find an N such that $\sup\{|f(z) - f_m(z)| \ : \ z \in A\} < \frac{\varepsilon}{3}$ for all $m > N$. Thus, $|f(z) - f_m(z)| < \frac{\varepsilon}{3}$ and $|f_m(\alpha) - f(\alpha)| < \frac{\varepsilon}{3}$ for all $m > N$. Finally, as f_m is continuous at α, we can find a δ such that $|f_m(z) - f_m(\alpha)| < \frac{\varepsilon}{3}$. Thus, if δ and N are chosen as above, then $|f(z) - f(\alpha)| < \varepsilon$ as required. □

Combining the previous two results, we obtain,

THEOREM 3.84. *Let $\sum a_n z^n$ be a power series and r be its radius of convergence. Let $f : D(0, r) \to \mathbb{C}$ be the function $f(z) = \sum a_n z^n$. Then f is continuous on $D(0, r)$.*

Proof. We will prove that if z_0 is an arbitrary point in $D(0,r)$ then f is continuous at z_0. As $z_0 \in D(0,r)$, $|z_0| < r$. Choose a ρ such that $|z_0| < \rho < r$. Then, by Theorem 3.80, we know the functions $f_m : D(0,\rho) \to \mathbb{C}$ defined as $f_m(z) = \sum_{n=1}^{m} a_n z^n$ converge uniformly to f on $D(0,\rho)$. As each f_m is continuous on $D(0,\rho)$, and f_m converge uniformly to f on $D(0,\rho)$, f is continuous on $D(0,\rho)$ by Theorem 3.83. As $z_0 \in D(0,\rho)$, f is continuous on z_0. As $z_0 \in D(0,r)$ was arbitrary, we have proved that f is continuous on $D(0,r)$. ☐

Example 3.85. Let $f_n : \mathbb{R} \to \mathbb{R}$ be the function defined as $f_n(x) = \frac{1}{n}\chi_{(0,n)}$. If $x \le 0$, the sequence $f_n(x)$ is the constant sequence containing just 0. Notice that

$$\sup\{|f_n(x) - f(x)| : x \in \mathbb{R}\} \le \frac{1}{n}.$$

Thus, the sequence f_n converge uniformly to f. However, as we saw earlier, $\int f_n$ does not converge to $\int f$. Thus, even the uniform limit is not too nice!

Definition 3.86 (Uniformly Cauchy). A sequence of functions $f_n : X \to \mathbb{R}$ is said to be **uniformly Cauchy** if for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $\forall x \in X$ and $\forall n, m > N$, $|f_n(x) - f_m(x)| < \varepsilon$.

Theorem 3.87. *A sequence of functions $f_n : A \to \mathbb{R}$ is uniformly Cauchy iff $f_n : A \to \mathbb{R}$ is uniformly convergent to some function $f : A \to \mathbb{R}$.*

Proof. Suppose f_n converges to f uniformly. Then given any ε, we can find an N such that

$$\sup\{|f(x) - f_n(x)| x \in A\} < \frac{\varepsilon}{2}$$

for all $n > N$. Moreover, if $x \in A$, then

$$|f(x) - f_n(x)| \le \sup\{|f(x) - f_n(x)| x \in A\} < \frac{\varepsilon}{2}.$$

Thus, if $n, m > N$, then by triangle inequality,

$$\begin{aligned} |f_n(x) - f_m(x)| &= |f_n(x) - f(x) + f(x) - f_m(x)| \\ &\le |f_n(x) - f(x)| + |f(x) - f_m(x)| \\ &\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for all $x \in A$.

Suppose f_n is uniformly Cauchy. Thus, in particular, the sequence $f_n(x)$ is Cauchy for all $x \in A$. Thus, by completeness of $\mathbb{R}$, $f_n(x)$ converges to some real number. Define a function $f : A \to \mathbb{R}$ where

$$f(x) = \lim_{n \to \infty} f_n(x).$$

We will show that f_n converges to f uniformly. Given any $\varepsilon > 0$, let N be a number such that $|f_n(x) - f_m(x)| < \frac{\varepsilon}{2}$ for all $n, m > N$. Given any x, let m be large enough that $|f_m(x) - f(x)| < \frac{\varepsilon}{2}$. Then,

$$\begin{aligned} |f_n(x) - f(x)| &= |f_n(x) - f_m(x) + f_m(x) - f(x)| \\ &\le |f_n(x) - f_m(x)| + |f_m(x) - f(x)| \\ &\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus, if $n > N$, then $|f_n(x) - f(x)| < \varepsilon$ for all $x \in A$. Thus, $\sup\{|f(x) - f_n(x)| x \in A\} < \varepsilon$ for all $n > N$. ☐

3.6.1. Devil's staircase.

Lemma 3.88.

$$sup\{|f_{n+1}(x) - f_n(x)| : x \in [0,1]\} \le \frac{1}{2} sup\{|f_n(x) - f_{n-1}(x)| : x \in [0,1]\}.$$

Proof. We will prove this using induction.

Base case - $n = 1$: We will first observe that $\sup\{|f_1(x) - f_0(x)| : x \in [0,1]\} = \frac{1}{6}$, by considering three cases.

$$\begin{aligned}\sup\left\{|f_1(x) - f_0(x)| : x \in \left[0, \frac{1}{3}\right]\right\} &= \sup\left\{\left|\frac{3x}{2} - x\right| : x \in \left[0, \frac{1}{3}\right]\right\} \\ &= \sup\left\{\left|\frac{x}{2}\right| : x \in \left[0, \frac{1}{3}\right]\right\} = \frac{1}{6}\end{aligned}$$

$$\sup\left\{|f_1(x) - f_0(x)| : x \in \left[\frac{1}{3}, \frac{2}{3}\right]\right\} = \sup\left\{\left|\frac{1}{2} - x\right| : x \in \left[\frac{1}{3}, \frac{2}{3}\right]\right\} = \frac{1}{6}$$

$$\begin{aligned}\sup\left\{|f_1(x) - f_0(x)| : x \in \left[\frac{2}{3}, 1\right]\right\} &= \sup\left\{\left|\frac{3x-1}{2} - x\right| : x \in \left[\frac{2}{3}, 1\right]\right\} \\ &= \sup\left\{\left|\frac{x-1}{2}\right| : x \in \left[\frac{2}{3}, 1\right]\right\} = \frac{1}{6}\end{aligned}$$

Now, we will use a similar analysis to show that $\sup\{|f_2(x) - f_1(x)| : x \in [0,1]\} = \frac{1}{12}$ which would complete the proof of the base case.

$$\begin{aligned}\sup\left\{|f_2(x) - f_1(x)| : x \in \left[0, \frac{1}{9}\right]\right\} &= \sup\left\{\left|\frac{f_1(3x)}{2} - \frac{3x}{2}\right| : x \in \left[0, \frac{1}{9}\right]\right\} \\ &= \sup\left\{\left|\frac{9x}{4} - \frac{3x}{2}\right| : x \in \left[0, \frac{1}{9}\right]\right\} \\ &= \sup\left\{\left|\frac{3x}{4}\right| : x \in \left[0, \frac{1}{9}\right]\right\} = \frac{1}{12}\end{aligned}$$

$$\begin{aligned}\sup\left\{|f_2(x) - f_1(x)| : x \in \left[\frac{1}{9}, \frac{2}{9}\right]\right\} &= \sup\left\{\left|\frac{f_1(3x)}{2} - \frac{3x}{2}\right| : x \in \left[\frac{1}{9}, \frac{2}{9}\right]\right\} \\ &= \sup\left\{\left|\frac{1}{4} - \frac{3x}{2}\right| : x \in \left[\frac{1}{9}, \frac{2}{9}\right]\right\} = \frac{1}{12}.\end{aligned}$$

$$\begin{aligned}\sup\left\{|f_2(x) - f_1(x)| : x \in \left[\frac{2}{9}, \frac{3}{9}\right]\right\} &= \sup\left\{\left|\frac{f_1(3x)}{2} - \frac{3x}{2}\right| : x \in \left[\frac{2}{9}, \frac{3}{9}\right]\right\} \\ &= \sup\left\{\left|\frac{9x-1}{4} - \frac{6x}{4}\right| : x \in \left[\frac{2}{9}, \frac{3}{9}\right]\right\} \\ &= \sup\left\{\left|\frac{3x-1}{4}\right| : x \in \left[\frac{2}{9}, \frac{3}{9}\right]\right\} = \frac{1}{12}\end{aligned}$$

$$\sup\left\{|f_2(x) - f_1(x)| : x \in \left[\frac{1}{3}, \frac{2}{3}\right]\right\} = \sup\left\{\left|\frac{1}{2} - \frac{1}{2}\right| : x \in \left[\frac{1}{3}, \frac{2}{3}\right]\right\} = 0.$$

$$\begin{aligned}\sup\left\{|f_2(x)-f_1(x)| : x\in\left[\frac{6}{9},\frac{7}{9}\right]\right\} &= \sup\left\{\left|\frac{f_1(3x-2)+1}{2}-\frac{f_0(3x-2)+1}{2}\right| : x\in\left[\frac{6}{9},\frac{7}{9}\right]\right\}\\ &= \sup\left\{\left|\frac{\frac{f_0(9x-6)}{2}+1}{2}-\frac{3x-1}{2}\right| : x\in\left[\frac{6}{9},\frac{7}{9}\right]\right\}\\ &= \sup\left\{\left|\frac{9x-6+2-6x+2}{4}\right| : x\in\left[\frac{6}{9},\frac{7}{9}\right]\right\}\\ &= \sup\left\{\left|\frac{3x-2}{4}\right| : x\in\left[\frac{6}{9},\frac{7}{9}\right]\right\} = \frac{1}{12}\end{aligned}$$

$$\begin{aligned}\sup\left\{|f_2(x)-f_1(x)| : x\in\left[\frac{7}{9},\frac{8}{9}\right]\right\} &= \sup\left\{\left|\frac{f_1(3x-2)+1}{2}-\frac{3x-2+1}{2}\right| : x\in\left[\frac{7}{9},\frac{8}{9}\right]\right\}\\ &= \sup\left\{\left|\frac{3}{4}-\frac{3x-1}{2}\right| : x\in\left[\frac{7}{9},\frac{8}{9}\right]\right\}\\ &= \sup\left\{\left|\frac{5-6x}{4}\right| : x\in\left[\frac{7}{9},\frac{8}{9}\right]\right\} = \frac{1}{12}\end{aligned}$$

$$\begin{aligned}\sup\left\{|f_2(x)-f_1(x)| : x\in\left[\frac{8}{9},\frac{9}{9}\right]\right\} &= \sup\left\{\left|\frac{f_1(3x-2)+1}{2}-\frac{3x-2+1}{2}\right| : x\in\left[\frac{8}{9},\frac{9}{9}\right]\right\}\\ &= \sup\left\{\left|\frac{\frac{3(3x-2)-2+1}{2}+1}{2}-\frac{3x-1}{2}\right| : x\in\left[\frac{8}{9},\frac{9}{9}\right]\right\}\\ &= \sup\left\{\left|\frac{3x-3}{4}\right| : x\in\left[\frac{8}{9},\frac{9}{9}\right]\right\} = \frac{1}{12}\end{aligned}$$

Induction hypothesis: If $[a,b]=[0,\frac{1}{3}]$ or $[\frac{1}{3},\frac{2}{3}]$ or $[\frac{2}{3},1]$, then

$$sup\{|f_{k+1}(x)-f_k(x)| : x\in[a,b]\} \le \frac{1}{2}sup\{|f_k(x)-f_{k-1}(x)| : x\in[a,b]\}.$$

Induction Step:

$$\begin{aligned}sup\left\{|f_{k+1}(x)-f_k(x)| : x\in\left[0,\frac{1}{3}\right]\right\} &= sup\left\{\left|\frac{f_k(3x)}{2}-\frac{f_{k-1}(3x)}{2}\right| : x\in\left[0,\frac{1}{3}\right]\right\}\\ &= \frac{1}{2}sup\left\{|f_k(3x)-f_{k-1}(3x)| : x\in\left[0,\frac{1}{3}\right]\right\}\\ &\le \frac{1}{4}sup\left\{|f_{k-1}(3x)-f_{k-2}(3x)| : x\in\left[0,\frac{1}{3}\right]\right\}\\ &= \frac{1}{2}sup\left\{|f_k(x)-f_{k-1}(x)| : x\in[a,b]\right\}\end{aligned}$$

□

Lemma 3.89.

$$sup\{|f_{n+1}(x)-f_n(x)| : x\in[0,1]\} \le \frac{1}{2^{n+1}}$$

Proof. Again we use a proof by induction. The previous lemma gives us the base case. Assume that the result is true for n, that is $sup\{|f_n(x)-f_{n-1}(x)| : x \in [0,1]\} \leq \frac{1}{2^n}$. Combining the previous lemma and the induction hypothesis, we have

$$sup\{|f_{n+1}(x) - f_n(x)| : x \in [0,1]\} \leq \frac{1}{2} sup\{|f_n(x) - f_{n-1}(x)| : x \in [0,1]\} \leq \frac{1}{2^{n+1}}$$

☐

Lemma 3.90. *The sequence f_n is uniformly cauchy and hence uniformly convergent.*

Proof. Let ε. As the series $\sum \frac{1}{2^k}$ converges, there exists an N such that if $n, m > N$ (we may assume without loss of generality that $n > m$), then $\sum_{k=m}^{n} \frac{1}{2^k} < \varepsilon$. Thus,

$$\begin{aligned}|f_n(x) - f_m(x)| &\leq |f_n(x) - f_{n-1}(x)| + \cdots + |f_{m+1}(x) - f_m(x)| \\ &\leq \frac{1}{2^n} + \cdots + \frac{1}{2^m} < \varepsilon.\end{aligned}$$

As x was arbitrary, we have our result. ☐

3.7. Analytic functions

Earlier, we saw that a power series defines a continuous function on its disk of convergence. It is natural to ask, if the functions corresponding to two power series agree on some neighbourhood of 0, are the two power series necessarily the same? First, we will state the question in a slightly more precise manner. Let $\sum a_n z^n$ and $\sum b_n z^n$ be two convergent power series such that their radius of convergence is greater than r and $\sum a_n z^n = \sum b_n z^n$ for all z with $|z| < r$, then is $a_n = b_n$ for all $n \in \mathbb{N}$? Subtracting one from the other, we get a power series $\sum c_n z^n = \sum (a_n - b_n) z^n = 0$ for all $|z| < r$. Thus, the earlier question is equivalent to the question. If $\sum c_n z^n$ is a convergent series with radius of convergence greater than r and let $\sum c_n z^n = 0$ for all z such that $|z| < r$, then is $c_n = 0$ for all $n \in \mathbb{N}$? The following theorem answers the question in the affirmative by telling us that a lot more can be guaranteed - if the function is zero at the origin (that is $a_0 = 0$) but the coefficients are not identically zero, you can find some neighbourhood of 0 with no additional zero in it.

Theorem 3.91. *Let $f(z) = \sum a_n z^n$ be a non-zero power series with a non-zero radius of convergence. If $f(0) = 0$, then there exists $s > 0$ such that $f(z) \neq 0$ if $0 < |z| < s$.*

Proof. As $f(0) = 0$, we can conclude that $a_0 = 0$. Let m be the smallest value for which $a_m \neq 0$. As $a_m \neq 0$, we have,

$$f(z) = \lim_{k\to\infty} \sum_{n=m}^{m+k} a_n z^n = \lim_{k\to\infty} \left(a_m z^m \sum_{n=0}^{k} \frac{a_{m+n}}{a_m} z^n \right).$$

For $0 < |z| < r$,

$$\lim_{k\to\infty} \left(a_m z^m \sum_{n=0}^{k} \frac{a_{m+n}}{a_m} z^n \right) \text{ and } \lim_{k\to\infty} \left(\frac{1}{a_m z^m} \right)$$

exist. Thus, by the property of limits, it must be that

$$g(z) = \sum_{n=0}^{\infty} \frac{a_{m+n}}{a_m} z^n.$$

converges. Thus, the radius of convergence of g is greater than or equal to r and $f(z) = a_m z^m g(z)$ when $|z| < r$. Further, for $n \geq 1$ define $b_n = \frac{a_{m+n}}{a_m}$ and $h(z) = \sum_{n=1}^{\infty} b_n z^n$. Then, $f(z) =$

$a_m z^m g(z) = a_m z^m(1 + h(z))$ when $|z| < r$. It is clear that $\sum_{n=1}^{\infty} b_n$ has a radius of convergence greater than or equal to r, just like g. Thus, h is continuous on $B(0, r)$ and $h(0) = 0$. Thus, there exists some $s > 0$ such that $|h(z)| < \frac{1}{2}$ for all $z \in B(0, s)$. That is, $|1 + h(z)| \geq \frac{1}{2}$ for all $z \in B(0, s)$. Moreover, $a_m z^m \neq 0$ if $z \neq 0$. Thus, $f(z) \neq 0$ for all $0 < |z| < s$. □

This theorem will come in handy in several situations. To give one example, the theorem allows us to prove

Theorem 3.92. *For every complex number z, $\cos^2(z) + \sin^2(z) = 1$.*

Proof. Follows from the above theorem and the fact that $\cos^2(x) + \sin^2(x) = 1$ for all real numbers x. □

Let us now get back to the main theme of our course - the two recipes. Recall Recipe 1, used the identification of $\mathbb{C}$ with $\mathbb{R}^2$ to construct functions from $\mathbb{C}$ to $\mathbb{C}$. And Recipe 2 used the structure in $\mathbb{C}$ to construct functions. Initially, we used the algebraic structure on $\mathbb{C}$ to construct polynomials (and rational functions). However, as this is a very restrictive class of functions, our idea was to use the topological structure of $\mathbb{C}$ to take limits of polynomials and enlarge the class of functions. This, led us to the study of formal power series and their convergence. We now know that distinct power series give rise to distinct continuous functions on their disks of convergence. However, it is still a bit unsatisfactory as their domains are somewhat restricted. Moreover, why should origin be given any special treatment - all points are created equal! Thus, we define,

Definition 3.93 (Analytic function). Let U be an open subset of $\mathbb{C}$. We say a function $f : U \to \mathbb{C}$ is analytic at a point z_0 if there exists a power series $\sum a_n(z - z_0)^n$ with a non-zero radius of convergence r and there exists some $0 < d < r$ such that $f(z) = \sum a_n(z - z_0)^n$ for all z such that $|z - z_0| < d$.

Moreover, by an analogue of Theorem 3.91 for power series at arbitrary points (which can be proved similarly, but with a bit more effort), we can see that this representation is in some sense unique.

We would now like to see some examples of analytic functions. Of course, the most obvious example should be a power series in its disk of convergence. Thankfully, the following theorem tells us that they are indeed analytic.

Theorem 3.94. *Suppose $f(z) = \sum a_n z^n$ be a power series whose radius of convergence is r. Then f is analytic on $D(0, r)$.*

Proof. To prove this theorem, we will pick an arbitrary point z_0 and show that there is a power series expansion $\sum b_n(z - z_0)^n$ that converges in some neighbourhood of z_0 and $\sum a_n z^n = \sum b_n(z - z_0)^n$. Notice,

$$\begin{aligned}\sum a_n z^n &= \sum a_n(z_0 + (z - z_0))^n \\ &= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \binom{n}{k} z_0^{n-k}(z - z_0)^k \\ &= \sum_{k=0}^{\infty} \left[\sum_{n=k}^{\infty} a_n \binom{n}{k} z_0^{n-k}\right](z - z_0)^k\end{aligned}$$

The last equality follows because the series is absolutely convergent and in an absolutely convergent series, rearrangement of terms do not affect the convergence of the series or its limit. From the above rearrangement, it is clear that

$$b_k = \sum_{n=k}^{\infty} a_n \binom{n}{k} z_0^{n-k}.$$

Moreover, $\sum b_k(z-z_0)^k$ converges wherever $\sum a_n z^n$ converges. Notice that the $B(z_0, s = r-|z_0|) \subset B(0,r)$ and so, the radius of convergence of $\sum b_k(z-z_0)^k$ should be greater than or equal to s.

□

We can also prove the following theorems about analytic function, from their counterparts for power series. We are not going to prove these theorems as their proofs are similar to the proofs we have encountered earlier.

THEOREM 3.95. *If f and g are analytic at z_0, $f+g$ is also analytic at z_0.*

THEOREM 3.96. *If f and g are analytic at z_0, fg is also analytic at z_0.*

THEOREM 3.97. *If f and g are analytic at z_0 and $g(z_0) \neq 0$, then f/g is also analytic at z_0.*

Going back to the two recipes, are the class of functions generated by the two recipes one and the same now. Recall that we had restricted our attention to differentiable functions from $\mathbb{R}^2$ to $\mathbb{R}^2$ in our Recipe 1. We have seen that functions defined by a power series are continuous functions. Thus, the same would be true of analytic functions as continuity is a local property. We can also show that analytic functions are differentiable.

THEOREM 3.98. *Let $f(z) = \sum a_n z^n$ be a power series with radius of convergence $r > 0$. Then, the series $\sum n a_n z^{n-1}$ has the same radius of convergence. And, the function $f : B(0,r) \to \mathbb{C}$ defined as $f(z) = \sum a_n z^n$ is differentiable on $B(0,r)$ and the derivative is given by the power series $\sum n a_n z^{n-1}$.*

Proof. We know that the radius of convergence of the series $\sum n a_n z^{n-1}$ is given by the formula

$$\limsup_{n\to\infty} \frac{1}{|na_n|^{\frac{1}{n}}}.$$

Given two sequences x_n, y_n if

$$\lim_{n\to\infty} x_n > 0,$$

then we can show that

$$\limsup_{n\to\infty} x_n y_n = \lim_{n\to\infty} x_n \limsup_{n\to\infty} y_n.$$

As

$$\lim_{n\to 0} n^{\frac{1}{n}} = 1 > 0$$

we have

$$\limsup_{n\to\infty} \frac{1}{|na_n|^{\frac{1}{n}}} = \left(\lim_{n\to\infty} \frac{1}{n^{\frac{1}{n}}}\right)\left(\limsup_{n\to\infty} \frac{1}{|a_n|^{\frac{1}{n}}}\right) = \limsup_{n\to\infty} \frac{1}{|a_n|^{\frac{1}{n}}}.$$

We can further use the binomial theorem to show that $\sum n a_n z^{n-1}$ is the derivative of $\sum a_n z^n$. However, I leave that as an exercise. □

Thus, all analytic functions are holomorphic. Are all holomorphic functions analytic? The following example might suggest that the answer should be no.

Example 3.99. The function

$$f(x) = \begin{cases} x^n \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

is differentiable $n-1$ times but not n times!

However, contrary to our expectation, every holomorphic function is analytic and understanding that would occupy a major part of the rest of the course. For now, we will focus on analytic functions, and prove various results about analytic functions.

Theorem 3.100 (Inverse function theorem). *Let $f(z) = a_1 z + \sum_{n=2}^{\infty} a_n z^n$ with $a_1 \neq 0$. Then, there exists a unique power series $g(z) = \sum_{n=1}^{\infty} b_n z^n$ such that $f(g(z)) = z = g(f(z))$. Moreover, if f is a convergent power series, then so is g.*

Proof. We can find this formal power series by assuming it has the form $\sum_{n=1}^{\infty} b_n z^n$ and comparing the coefficients of z^n on the two sides of $f(g(z)) = z$. For example, by comparing the coefficients of z we immediately see that $a_1 b_1 = 1$ or $b_1 = \frac{1}{a_1}$. Similarly, comparing the coefficients of z^2, we see $a_1 b_2 - a_2 b_1^2 = 0$. Thus, $b_2 = \frac{a_2 b_1^2}{a_1}$. Similarly, comparing the cubic terms of $f(g(z))$, we see that $a_1 b_3 - 2a_2 b_1 b_2 + a_3 b_1^3 = 0$. Thus,

$$b_3 = \frac{2a_2 b_1 b_2 - a_3 b_1^3}{a_1}.$$

Similarly, find b_4, and b_5 and convince yourself that you can similarly find all b_n. Or more precisely, you can use induction to prove that

$$b_n = \frac{1}{a_1} P_n(a_2, \ldots, a_n, b_1, \ldots, b_{n-1})$$

where P_n is a polynomial with positive integer coefficients. Moreover, as $a_1 b_1 = 1$, b_1 is not equal to 0. Thus, there exists a power series h such that $g(h(z)) = z$. Thus,

$$g(f(z)) = g(f(g(h(z)))) = g(f \circ g(h(z))) = g(h(z)) = z.$$

Thus, $g(f(z)) = z = f(g(z))$. That is a formal inverse exists and is unique.

Now, we will prove that the radius of convergence of g is non-zero. To simplify notation, we first prove that we may assume $a_1 = 1$. As $a_1 \neq 0$, you can consider the function $F(z) = \frac{f(z)}{a_1}$. If $h(z) = \frac{z}{a}$, then $F(z) = h \circ f(z)$. Thus, the inverse of F denoted by G is the function $G(z) = g(a_1 z)$. Thus, $G(z) = \sum b_n (a_1 z)^n = \sum b_n (a_1)^n z^n$. Thus, if r_G is the radius of convergence of G and r_g is the radius of convergence of g, then

$$\begin{aligned} r_G &= \frac{1}{\limsup_{n\to\infty} |a_1^n b_n|^{\frac{1}{n}}} = \frac{1}{\limsup_{n\to\infty} |a_1||b_n|^{\frac{1}{n}}} \\ &= \frac{1}{|a_1|} \frac{1}{\limsup_{n\to\infty} |b_n|^{\frac{1}{n}}} = \frac{r_g}{|a_1|} \end{aligned}$$

Thus, G is convergent in some disk iff g is convergent in some disk. Hence, it is enough to prove the case when $a_1 = 1$.

Let

$$F(z) = z - \sum_{n=2}^{\infty} A_n z^n$$

be a power series such that $A_n \in \mathbb{R}$, $A_n \geq 0$, $|a_n| \leq A_n$ for all n. Let $G(z) = \sum_{n=1}^{\infty} B_n$ be the formal inverse of F. Then $B_1 = 1$ and

$$B_n = P_n(A_2, \dots A_n, B_1, \dots B_{n-1})$$

where P_n is the same polynomial as before (note that a_1 is assumed to be 1). Thus, we can recursively prove that $|b_n| \leq B_n$. Thus, it suffices to find a nice F such that its formal inverse G has a positive radius of convergence.

As f is a convergent power series, $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \frac{1}{r} \neq \infty$. Thus, if $K > \frac{1}{r}$ then there exists N such that $|a_n|^{\frac{1}{n}} < K$ for all $n > N$. In other words, $|a_n| < K^n$ for all $n > N$. Now choose $A > \max a_1, |a_2|^{\frac{1}{2}}, \dots, |a_N|^{\frac{1}{N}}, K$. Then, $|a_n| A^n$ for all n. Finally, define

$$F(z) = z - \sum_{n=2}^{\infty} A^n z^n = z - \frac{A^2 z^2}{1 - Az}.$$

As $F(G(z)) = z$, we get

$$G(z) - \frac{A^2 G(z)^2}{1 - AG(z)} = z.$$

That is,

$$(A^2 + A)G(z)^2 - (1 + Az)G(z) + z = 0.$$

Thus,

$$G(z) = \frac{1 + Az - \sqrt{(1 + Az)^2 - 4z(A^2 + A)}}{2(A^2 + A)} = \frac{1 + Az - \sqrt{(1 + Az)^2 \left[1 - \frac{4z(A^2+A)}{(1+Az)^2}\right]}}{2(A^2 + A)}.$$

We can now use Exercise 12 to find $\sqrt{(1 + Az)^2 \left[1 - \frac{4z(A^2+A)}{(1+Az)^2}\right]}$ which is a convergent power series. Moreover, $\frac{1}{2(A^2+A)}$ is also a convergent power series. Thus, by algebra of power series, we know that G is a convergent power series. □

Remark 3.101. Notice that $a_1 = f'(z)$. Thus, the theorem is analogous to the inverse function theorem you learnt in Real analysis.

DEFINITION 3.102 (Open map). A function $f : U \to \mathbb{C}$ is called an open map, if given any open subset V of U, its image $f(U)$ is also open.

THEOREM 3.103 (Open mapping theorem). *Let f be analytic on an open set U and assume that given $z \in U$, there exists a neighbourhood $B(z, \varepsilon)$ such that f is not constant on $B(z, \varepsilon)$. Then, f is an open map.*

Proof. Let $f(z) = \sum a_n z^n$ be a power series and let m be the smallest natural number such that $a_m \neq 0$. Then just as in Theorem 3.91, we can prove that $f(z) = a_m z^m (1 + h(z))$. In Exercise 12, you will prove that you can take the m-th root of $1 + h(z)$ to get an analytic function $1 + h_1(z)$. Thus, if $f_1(z) = az(1 + h_1(z))$ then $f(z) = f_1(z)^m$. Notice that the function $z \to z^m$ takes a disk around 0 to a disk around 0. Thus, if we can show that given an open set U containing 0, $f_1(U)$ contains a disk around 0, then $f(U)$ will also contain a disk around 0. However, as f_1 satisfies the hypothesis of Theorem 3.100, we have the result. Applying this trick to the power series expansion of f at all points gives the more general result. □

The open mapping theorem also helps us prove another important theorem called the local maximum modulus principle.

THEOREM 3.104. *Let f be analytic on a open set U. Let $z_0 \in U$ be such that $|f(z_0)| \geq |f(z)|$ for all $z \in U$. Then, there exists some open subset D of U such that f is constant on D.*

Proof. Let the power series expansion of f at z_0 be $\sum a_n(z-z_0)^n$. We will prove this theorem using a proof by contradiction. Assume f is not the constant $a_0 = f(z_0)$, then by the open mapping theorem, there exists a neighbourhood $B(z_0, \varepsilon)$ such that $f(B(z_0, \varepsilon))$ is an open set. And therefore, there exists some δ such that the ball $B(f(z_0), \delta) \subset f(U)$. But, the ball $B(f(z_0), \delta)$ has elements whose modulus is greater than $f(z_0)$. This contradicts the assumption that $|f(z_0)| \geq |f(z)|$ for all $z \in U$. □

And the local maximum modulus principle allows us to prove the fundamental theorem of algebra.

THEOREM 3.105. *If* $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ *be a polynomial with* $n > 0$ *and* $a_n \neq 0$*, then there exists some complex number* z_0 *such that* $f(z_0) = 0$.

Recall from high-school algebra that if z_0 is a zero of a polynomial, then $z - z_0$ divides the polynomial. By recursively applying the result on the quotient, we have the famous fundamental theorem of algebra.

Proof. Once again we will use a proof by contradiction. Assume the polynomial has no zeroes. Then the function $1/f$ is analytic on $\mathbb{C}$. Notice,

$$\lim_{|z|\to\infty} \frac{a_n z^n}{a_0 + a_1 z + \cdots + a_n z^n} = 1.$$

That is the polynomial $f(z)$ behaves like $a_n z^n$ for large values of $|z|$. And it is clear that as $|z|$ goes to ∞, $1/(a_n z^n)$ converge to 0. Therefore. $1/f(z)$ should also converge to 0. Let z_0 be some complex number such that $f(z_0) \neq 0$. Choose a positive real number R such that $|z_0| < R$ and if $|z| \geq R$, then

$$\frac{1}{|f(z)|} < \frac{1}{|f(z_0)|}.$$

As $1/f$ is a continuous function (as it is analytic) on the $\overline{B(0,R)}$ (the ball with its boundary points), it has a maximum at some point in $\overline{B(0,R)}$, say z_1. As $\frac{1}{|f(z)|} < \frac{1}{|f(z_0)|}$ for all $|z| \geq R$, the point z_1 cannot lie on the boundary. By, the local maximum modulus principle, we conclude there is a neighbourhood $B(z_1, \delta)$ where $1/f$ is constant. This would imply that f is also constant in $B(z_1, \delta)$, which is impossible.

□

Exercises

(1) Find the first 3 terms in the power series expansion of $\frac{z^2}{z+2}$.

(2) Use the power series for $\sin(z)$ and the power series for $\frac{1}{\cos(z)}$ above to find the first 3 terms in the power series of $\tan(z)$.

(3) Show that if $\sum z_n$ converge, then z_n converge to 0.

(4) Given two series of complex numbers $A = \sum a_n$ and $B = \sum b_n$ that converge, show that the sum $A + B$ also converge.

(5) Given two series of complex numbers $A = \sum a_n$ and $B = \sum b_n$ that converge, show that the product AB defined as

$$AB = \sum_{i=0}^{\infty} \left(\sum_{k=0}^{i} a_k b_{i-k} \right)$$

also converge.

(6) Find the terms of order less than or equal to 3 in the following:

  (a) $\frac{1}{1+z+z^2}$
  (b) $\frac{1}{\cos(z)}$
  (c) $\frac{\sin(z)}{\cos(z)}$
  (d) $\frac{z}{(z-1)^2}$

(7) Give an example of
  (a) a power series whose radius of convergence is equal to 2.
  (b) a power series whose radius of convergence is equal to α where α is a real number.
  (c) a power series whose radius of convergence is 1 and such that the corresponding function is continuous on the closed unit disc. [This question is from Complex Analysis by Serge Lang]
  Hint: Use $\sum z^n/n^p$ for a suitable p. If you struggle for a choice of p, check Lang's book ;)
  (d) two power series $\sum a_n z^n$ and $\sum b_n z^n$ such that the product has a radius of convergence greater than the radius of convergence of one of the two power series.

(8) Determine the radius of convergence of the following power series:

  (a) $\sum \alpha^n z^n$
  (b) $\sum n^n z^n$
  (c) $\sum \frac{1}{n!} z^n$
  (d) $\sum \frac{(n!)^3}{3n!}$

(9) Show that the sequence f_n from Example 3.68, Example 3.70, Example 3.71, and Example 3.72 do not converge uniformly.

(10) Let $\mathcal{B}$ be the set of bounded function. Then, show that the function $d_{\max} : \mathcal{B} \times \mathcal{B} \to \mathbb{R}$ defined as $d_{\max}(f,g) = \sup\{|f(x) - g(x)| : x \in \mathbb{R}\}$ is a metric on $\mathcal{B}$.

(11) Define X to be the collection of all functions $f : \mathbb{R} \to \mathbb{R}$ such that $\int_{-\infty}^{\infty} |f(x)| dx < \infty$ and $\sim$ to be the relation $f \sim g$ if $\int |f - g| = 0$.
  (a) Show that $\sim$ is an equivalence relation.
  (b) Let $\mathcal{L}^1$ be $X/\sim$. Show that $d_{avg}(f,g) = \int_{-\infty}^{\infty} |f(x) - g(x)| dx$ is a metric on $\mathcal{L}^1$

(12) Finding the q^{th} root using binomial series:
  (a) Show that the radius of convergence of the binomial series is equal to 1 if α is not equal to an integer greater than or equal to 0.
  (b) By multiplying the two power series, deduce the relation $B_n(z)B_m(z) = \sum_{r=0}^{\infty} C_r(n,m) z^r$ where $C_r(n,m) = \sum_{j=0}^{r} \binom{n}{j}\binom{m}{r-j}$.
  (c) Recall $B_n(z) = (1+z)^n$ if n is a natural number. Use it to prove $B_n(z)B_m(z) = B_{n+m}(z)$.
  (d) Deduce that $C_r(n,m) = \binom{n+m}{m}$ when m and n are positive integers. But, $C_r(n,m)$ and $\binom{n+m}{m}$ are polynomials in n and m and they agree on infinitely many values of m and n implies they have to be equal for all complex values of n and m.
  (e) Deduce that $\left(B_{\frac{1}{q}}(z)\right)^q = (1+z)$ for all integers q. In other words, you should think of $B_{\frac{1}{q}}(z)$ as $(1+z)^{\frac{1}{q}}$.

(13) Show that the sequence of function $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$ converge uniformly to $f(x) = |x|$. Further show that f_n is differentiable on $\mathbb{R}$ but f is not differentiable at 0. Thus, uniform limit of differentiable functions need not be differentiable.

Remark 3.106. Recall that we proved the continuity of a power series on its disk of convergence(and thus of analytic functions) using the observation that uniform limits of continuous functions are continuous. The above example suggests we cannot use a similar argument for the proof of differentiability of a power series in its disk of convergence.

Line Integrals and Cauchy's theorem

Let U be an open set in $\mathbb{C}$ and $f : U \to \mathbb{C}$ be a (continuous) function. We define the integral of f along a differentiable curve $\gamma : [a, b] \to \mathbb{C}$ to be

$$\int_\gamma f = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

If we assume f is "reasonably nice" then the integral on the right hand side in the above equation is well defined. In analysis, you might have studied that the continuity of f is a sufficient condition.

Example 4.1. Let $f : \mathbb{C} \to \mathbb{C}$ be the function $f(z) = z$ and let $\gamma : [0, 1] \to \mathbb{C}$ be the function $\gamma(t) = t + it$. Then, $\gamma'(t) = 1 + i$ and

$$\int_\gamma f = \int_0^1 (t + it)(1 + i) = \int_0^1 0dt + i\int_0^1 2tdt = 2i.$$

Definition 4.2. We say a curve $\gamma : [a, b] \to \mathbb{C}$ is a reparametrisation of a curve $\beta : [c, d] \to \mathbb{C}$ if there exists a differentiable function $\varphi : [a, b] \to [c, d]$ such that $\gamma(t) = \beta(\varphi(t))$.

Example 4.3. The curve $\gamma : [0, 1] \to \mathbb{C}$ defined as $\gamma(t) = t + it$ is a reparametrisation of the curve $\beta : [0, 2] \to \mathbb{C}$ defined as $\beta(t) = \frac{t}{2} + i\frac{t}{2}$. Define $\varphi : [0, 1] \to [0, 2]$ as $\varphi(t) = 2t$ and $\gamma(t) = \beta(\varphi(t))$.

Theorem 4.4. *The integral of a function is independent of the parametrisation. More precisely, if $\gamma : [a, b] \to \mathbb{C}$ is a reparametrisation of $\beta : [c, d] \to \mathbb{C}$ and $f : U \to \mathbb{C}$ is a function, then*

$$\int_\gamma f = \int_\beta f.$$

Proof. Using Chain Rule, we see that

$$\int_\gamma f = \int_a^b f(\gamma(t))\gamma'(t)dt = \int_a^b f(\beta(\varphi(t)))\beta\circ\varphi'(t)dt = \int_c^d f(\beta(s))\beta'(s)ds = \int_\beta f$$

□

We can easily generalise the definition to piece-wise differentiable curves. Let U be an open set in $\mathbb{C}$ and $f : U \to \mathbb{C}$ be a continuous function. Let $\gamma : [a, b] \to \mathbb{C}$ be a piece-wise differentiable curve. That is there exists $a = a_0 < a_1 < \cdots < a_n = b$ be such that $\gamma_i = \gamma|_{[a_i, a_i+1]}$ is differentiable. Then

$$\int_\gamma f = \sum_{i=0}^{n-1} \left(\int_{\gamma_i} f \right) = \sum_{i=0}^{n-1} \left(\int_{a_i}^{a_{i+1}} f(\gamma(t))\gamma'(t)dt \right).$$

Definition 4.5. A function $g : U \to \mathbb{C}$ is a called a primitive or anti-derivative of the function $f : U \to \mathbb{C}$ if $g'(z) = f(z)$ for all $z \in U$.

Theorem 4.6. *Let* $f : U \to \mathbb{C}$ *be a continuous function such that* f *h*

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