



Characterising Continuous Functions

By the Properties of their Graphs

by

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Abstract

The intuition for a ‘continuous function’ is often of drawing something with a pen without lifting it. What’s being drawn is the graph of the function, and not lifting is similar to the idea of connectedness in topology. However, the property of connectedness is not strong enough as a condition for continuity, as illustrated by the existence of the topologist’s sine curve. Connectedness of the graph and related ideas - such as not connected or locally connected - lead to varied results about the continuity of its function. By reading the PhD thesis ‘Closed and connected graphs of functions; examples of connected punctiform spaces’, authored by Michal Ryszrd Wojcik, we’ve studied and explored such relationships. We began by restricting ourselves to ‘nice’ domains like the real numbers and were able to characterise continuous functions by the graph, before moving on further to general topological spaces, or at least spaces with fewer restrictions.

Dedication

To Amma and Appa.

Declaration

I, Tanvi Chaubey, declare that the B.Sc. (Honours) in Mathematics thesis titled “Characterising Continuous Functions” contains no material that has been accepted for the award of any other degree or diploma in my name, in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text. In addition, I certify that no part of this work will, in the future, be used in a submission in my name, for any other degree or diploma in any university or other tertiary institution without the prior approval of Azim Premji University.

Acknowledgements

As an introduction to research, I really enjoyed working on the project and the hours spent going through problems with my mentor, Professor Divakaran. Although I stuttered at times, he was always willing to spend extra time on meetings over the week to work through one more proof.

Thank you to my parents for long calls and longer updates on the progress of the thesis and life in general. I can forever count on them to be in my corner and listen.

I'd like to acknowledge my support system throughout these last three years – friends made both in the classroom working through honours, and out of it. In alphabetical order: Anmol, Charu, Michelle, Mugdha, Naina, Saanvi, Vaishnavi, Vedika, Vijaylakshmi and Bezal, Chinmayee, Daniel, Kashish, Nishanth, Tulasi, Varsha. Thank you to all, I would not have made it through this without them!

Regards to Michal Ryszard Wojcik, the author of the paper this thesis is based on. The results he found were beautiful and I loved puzzling through them.

Finally thank you to the math faculty at Azim Premji University for fostering my interest in pure mathematics, leading me to take up a project like this.

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Chapter 1

Introduction

Intuitively, we think of a continuous function as being able to draw a line without ever having to lift the pen. Yet what is being drawn is not truly the function, but a representation of its *graph*. The idea of not lifting the pen i.e. the graph never being ‘broken’, is called connectedness. So, is a connected graph enough to ensure that any function is continuous? It is fair to come to this conclusion, since every continuous function has a connected graph.

Unfortunately, this is not the case. The topologist’s sine curve: $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ defined by $\Phi(x) = \sin(\frac{1}{x})$ for non-zero x and 0 at $x = 0$,

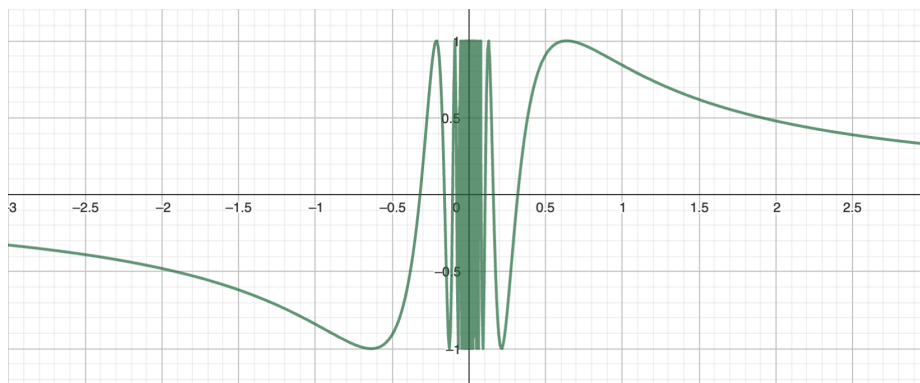


Figure 1.1: The graph of the Φ function.

(although visually might seem unlikely) has a connected graph! However, it is not a continuous function.

We will look at the proof of this claim.

For the undergraduate reader to understand this proof in its entirety, we shall first define terms used to describe the ‘standard topology’ on \mathbb{R}^2 , the space we are working in. In the second chapter we will further explore necessary definitions and lemmas with supporting examples.

Definition 1. Let $U \subset \mathbb{R}^2$. We say U is an open subset of \mathbb{R}^2 if for any $(x, y) \in U$, $\exists \epsilon > 0$ such that $B_\epsilon(x, y) := \{(x', y') \mid \sqrt{(x - x')^2 + (y - y')^2} < \epsilon\} \subset U$.

Definition 2. Let $K \subset \mathbb{R}^2$. We say K is connected if there do not exist disjoint non-empty open $U, V \subset K$ such that $U \cup V = K$.

Open subsets of K are of the form $K \cap U'$ where U' is open in \mathbb{R}^2 . To read more on this, refer to Example 4 on subspace topology in the next chapter.

Claim 1. The topologist’s sine curve has a connected graph.

Proof. Suppose the graph of Φ is not connected. Then, there exists non-empty clopen (closed and open) $U \subsetneq Gr(\Phi)$. Clearly $V = U^c$ is also a non-trivial clopen subset of $Gr(\Phi)$. Without loss of generality, $(0, 0) \in U$. Since U is open, $\exists \epsilon > 0$ s.t. $Gr(\Phi) \cap B_\epsilon((0, 0)) \subset U$. Then, for large enough n ($n > \frac{1}{2\pi\epsilon}$), the points $(\frac{1}{2n\pi}, 0)$ and $(-\frac{1}{2n\pi}, 0)$ belong to $Gr(\Phi) \cap B_\epsilon((0, 0))$.

We can show that if $(a, \sin(\frac{1}{a})) \in U$, then for every x where $\text{sign}(x) = \text{sign}(a)$, $(x, \sin(\frac{1}{x})) \in U$. Without loss of generality, assume $a > 0$ and let $A := \{x \mid (y, \sin(\frac{1}{y})) \in U, \forall y \in [a, x]\}$. If A is bounded above, by the Completeness Axiom a least upper bound exists. Let $b = \sup(A)$. Since U is closed, $(b, \sin(\frac{1}{b})) \in U$. By the openness of U , $\exists \epsilon_b > 0$ s.t. $\forall x \in (b - \epsilon_b, b + \epsilon_b)$, $(x, \sin(\frac{1}{x})) \in U$. Clearly $b + \epsilon \in A$. Yet, $b + \epsilon > b = \sup(A)$. This is a contradiction, so A does not have an upper bound. Therefore $(x, \sin(\frac{1}{x})) \in U, \forall x > a$. Similarly, let $B := \{x \geq 0 \mid (y, \sin(\frac{1}{y})) \in U, \forall y \in [x, a]\}$. B is bounded below by 0. If $\inf(B) > 0$, we get a contradiction similar to above. Therefore, if $a > 0$ and $(a, \sin(\frac{1}{a})) \in U$, then $(x, \sin(\frac{1}{x})) \in U, \forall x > 0$.

Using a parallel argument, if $a < 0$ and $(a, \sin(\frac{1}{a})) \in U$, then $(x, \sin(\frac{1}{x})) \in U, \forall x < 0$. Since

we have found two such points, we use this implication to get $Gr(\Phi) = \{(x, \sin(\frac{1}{x})) \mid x < 0\} \cup \{(0, 0)\} \cup \{(x, \sin(\frac{1}{x})) \mid x > 0\} = U$. We reach a contradiction since U is a proper subset of $Gr(\Phi)$. Therefore, the graph is connected. \square

Claim 2. *The topologist's sine curve is not continuous.*

Proof. Suppose Φ is continuous. Then, by the sequential definition of continuity: if $(x_n) \rightarrow x$, then $(f(x_n)) \rightarrow f(x)$. Let $x = 0$. Suppose $x_n = \frac{2}{\pi(1+4n)}$. Then $f(x_n) = \sin \frac{1}{x_n} = \sin(\frac{\pi}{2} + 2n\pi) = 1, \forall n \in \mathbb{N}$. This is the constant sequence 1, which converges to $1 \neq 0 = f(x)$. Therefore, f is not a continuous function. To note: 'approaching' 0 from *any* line other than $y = 0$ (i.e. not $x_n = \frac{1}{2n\pi}$) would also result in a non-zero constant sequence. \square

If we strengthened the properties of the graph, could we ensure continuity? What is missing?

When we 'zoom' in on the graph, we notice that locally, it looks like disconnected lines. In fact, this is close to what is actually happening.

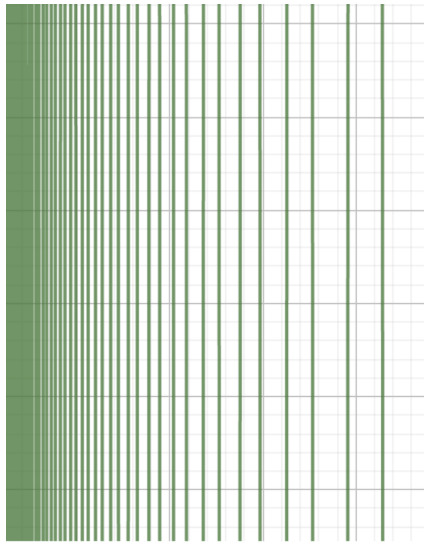


Figure 1.2: A zoom in on our graph.

Definition 3. *Let $U \subset \mathbb{R}^2$ be an open set and $x \in U$. We say U is locally connected at x if there exists open connected $K \subset \mathbb{R}^2$ such that $x \in K \subset U$.*

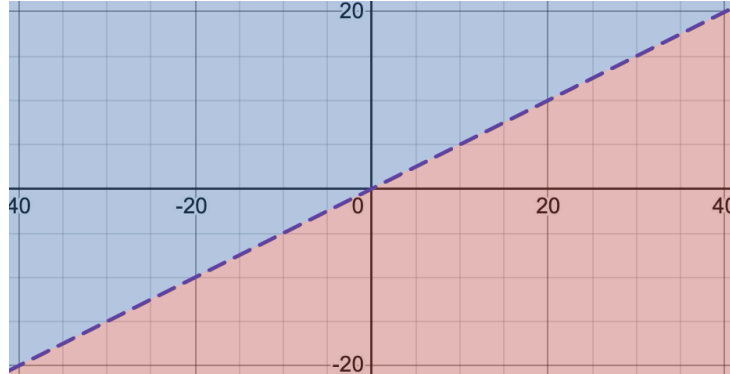
Claim 3. *The graph of the topologist's sine curve is not locally connected.*

Proof. We will show that $Gr(\Phi)$ is not locally connected at $(0,0)$. More precisely, we will construct a separation for $Gr(\Phi) \cap B_\epsilon((0,0))$, $\forall \epsilon > 0$.

Let $\epsilon > 0$. Then we can find n such that $\frac{1}{2n\pi} < \epsilon$. For this x , $(\frac{1}{2n\pi}, \sin(\frac{1}{2n\pi})) = (\frac{1}{2n\pi}, 0) \in B_\epsilon((0,0))$. We define $m := \sup\{x < \frac{1}{2n\pi} \mid (x, \sin(\frac{1}{x})) \notin B_\epsilon((0,0))\}$ and $M := \inf\{x > \frac{1}{2n\pi} \mid (x, \sin(\frac{1}{x})) \notin B_\epsilon((0,0))\}$.

The set $V_1 = \{(x, \sin(\frac{1}{x})) \mid x \in (m, M)\}$ is open in $Gr(\Phi) \cap B_\epsilon((0,0))$ since $V_1 = ((m, M) \times \mathbb{R}) \cap (Gr(\Phi) \cap B_\epsilon((0,0)))$. This follows since Φ is a well-defined function. V_1 is also closed since its complement V_2 equals $((-\epsilon, m) \times \mathbb{R}) \cup ((M, \epsilon) \times \mathbb{R}) \cap (Gr(\Phi) \cap B_\epsilon((0,0)))$. Together, V_1 and V_2 form a separation of $Gr(\Phi) \cap B_\epsilon((0,0))$. Therefore, $Gr(\Phi)$ is not locally connected. \square

Suppose instead we remove the graph from the plane. If it is connected, it is possible that removing the graph would disconnect the space, such as for the continuous function $f(x) = \frac{x}{2}$:



Think of cutting a block of clay using wire - removing this line turns \mathbb{R}^2 into two disjoint half-planes. This is the complement of the graph of f .

Claim 4. *The complement of the graph of the topologist's sine curve is connected.*

Proof. Suppose the complement is not connected. Then $\exists U, V \subset \mathbb{R}^2 \setminus Gr(\Phi)$ s.t. $U \neq \emptyset \neq V$, $U \cap V = \emptyset$ and $U \cup V = \mathbb{R}^2 \setminus Gr(\Phi)$.

Without loss of generality, let $(0, \frac{1}{2}) \in U$. We will show that $\exists(x, y) \in U$ s.t. $x > 0$ and for some $\epsilon > 0$, $y = \sin(\frac{1}{x}) + \epsilon$. Intuitively, this is a point in the complement slightly to the right of our original point and 'above' the graph. Since U is open, we can find δ s.t.

$B_\delta((0, \frac{1}{2})) \cap (\mathbb{R}^2 \setminus Gr(\Phi)) \subset U$. Let $x = \frac{\delta}{2}$. Then $d_E((\frac{\delta}{2}, y), (0, \frac{1}{2}))$ must be less than δ . Now we must show that such a $y \in (\sin(\frac{1}{\delta/2}), \frac{1}{2} + \delta \frac{\sqrt{3}}{2})$ exists.

For large enough n , we can find $\delta' \leq \delta$ (take $\delta' = \frac{6}{\pi(1+12n)}$) s.t. $\sin(\frac{1}{\delta'/2}) = \frac{1}{2}$. Clearly, $\sin(\frac{2}{\delta'}) < \frac{1}{2} + \delta' \frac{\sqrt{3}}{2}$. Then $x = \frac{\delta'}{2}$ and $\epsilon = \delta' \frac{\sqrt{3}}{2}$ i.e. $(\frac{\delta'}{2}, \sin(\frac{1}{\delta'/2}) + \delta' \frac{\sqrt{3}}{2}) \in U$. Now we can show that $\{(x, y) \mid x > 0, y > f(x)\} \subset U$. Consider $A = \{y \mid (x, y) \in U, y > f(x) \forall x > 0\}$. Since this set is non-empty, suppose it is bounded above. Let $b = \sup(A)$. Since U is closed, $(x, b) \in U, \forall x > 0$. By the openness of U , $\exists \epsilon_b > 0$ s.t. $\forall y \in (b - \epsilon_b, b + \epsilon_b), (x, y) \in U$. Then $b + \epsilon \in A$. This is a contradiction, so the set is not bounded above (it is bounded below by the graph).

We use a similar argument for the other ‘quadrants’ of the complement, and get that $\mathbb{R}^2 \setminus Gr(\Phi) \subset U$. We reach a contradiction since U is a proper subset of $Gr(\Phi)$. Therefore, the complement of the graph is connected. □

In the first section of the third chapter, we will explore whether adding local connectedness to our graph properties ensures that the function is continuous, and when it fails. In the third, we will take a look at the complement of a connected graph and what it gives us for continuity.

It turns out that the topologist’s sine curve does not have a closed graph, because of its behaviour around zero.

Claim 5. *The graph of the topologist’s sine curve is not closed.*

Proof. We will use a similar proof to the proof for claim 2. Suppose $Gr(\Phi)$ is closed. Then $(x_n, f(x_n)) \rightarrow (x, f(x))$. However, if $x_n = \frac{2}{\pi(1+4n)}$, then $(x_n, y) = (\frac{2}{\pi(1+4n)}, \sin(\frac{\pi}{2} + 2n\pi)) = (\frac{2}{\pi(1+4n)}, 1) \rightarrow (0, 1) \neq (0, 0) = (0, f(0))$. Therefore, $Gr(\Phi)$ is not closed. □

In the fourth chapter, we will understand whether or not adding closed to our graph properties ensures that the function is continuous, and when it fails. In particular, we will focus

on certain *sections* of the function and understand its continuity from how well-behaved the function on these sections and the spaces we are working with are.

In these chapters we will see that for domains like \mathbb{R} , adding any of the properties above ensures continuity. So the domain and codomain of the function greatly matters.

For higher and even slightly more complex spaces, these properties are still not sufficient. We will explore such examples of functions with very ‘nice’ graphs that still yet fail to be continuous, in the fifth chapter.

To understand concepts in topology, I have referenced Conway [2014](#), Kumaresan [2005](#), Munkres [n.d.](#) The book I followed was Essential Topology, Crossley [2006](#).

All results have been taken from the PhD thesis: Wójcik [2008](#) – Closed and connected graphs of functions; examples of connected punctiform spaces, authored by Michal Ryszard Wojcik, University of Silesia in Katowice. Only chapter 4 (titled Punctiform Spaces as Connected Graphs of Functions) of the paper will not be covered in this thesis.

Chapter 2

Preliminaries

For even a first read of the PhD thesis, extensive reading to understand concepts in the mathematical field of topology was necessary. In order to make reading this thesis more accessible, this chapter delves into every definition and lemma used in proofs in later chapters. With the topological definitions, supporting examples and pictures are provided. Where possible, proofs or proof outlines of stated lemmas have been included.

2.1 Definitions

Definition 1. Let $f : X \rightarrow Y$. We define the graph of f as $Gr(f) := \{(x, f(x)) \mid x \in X\}$.

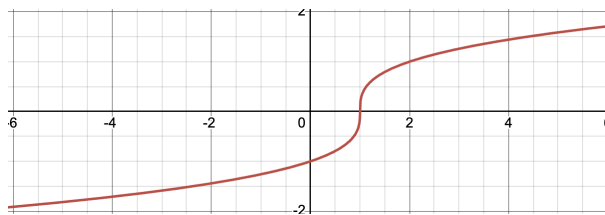
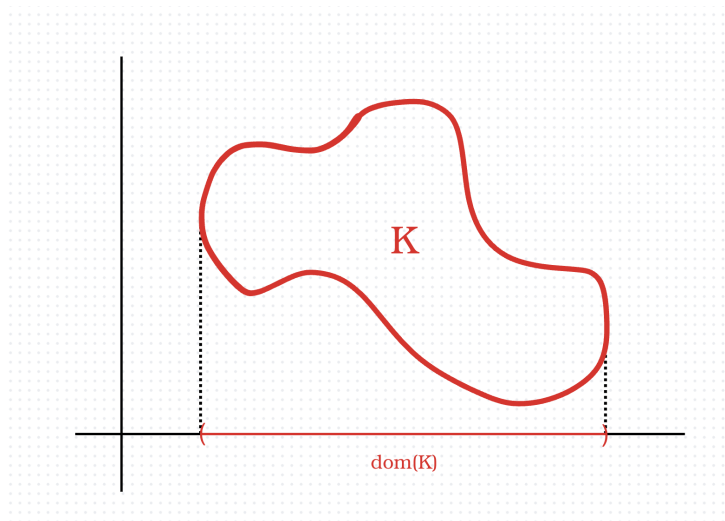


Figure 2.1: The graph of $f(x) = (x - 1)^{\frac{1}{3}}$

Definition 2. Let $K \subset X \times Y$. We define the domain of K as $dom(K) := \{x \in X \mid (x, y) \in K\}$.

Example 1. In figure the figure, the domain of K is the interval marked in red.



Definition 3. Let X be a non-empty set. A set τ of subsets of X is a topology on X if and only if:

- $\emptyset, X \in \tau$.
- arbitrary (i.e. finite or infinite) union of elements in τ belong to τ .
- finite intersection of elements in τ belong to τ .

Although multitudes of topologies have been discovered, we will only use the ones mentioned in this chapter.

Definition 4. We define a topological space as the pair (X, τ) , where the set X is equipped with the topology τ .

Note: for ease of notation, while writing we may replace (X, τ) with X as a topological space.

Example 2. Let X be a topological space. We define the indiscrete topology as $\tau = \{\emptyset, X\}$.

Example 3. Let X be a topological space. We define the discrete topology as $\tau = \mathcal{P}(X)$.

These topologies can be defined on any set, they are trivial examples.

Example 4. Let X be a topological space. Suppose $J \subset X$. Then, we define the subspace topology τ' as the topology on J , induced by the topology on X . Every element of τ' is of the form $J \cap U$, where $U \in \tau$.

Definition 5. Let X be a non-empty set. A set \mathcal{B} of subsets of X is called a basis on X if and only if:

- for every $x \in X$, we can find $B \in \mathcal{B}$ such that $x \in B$
- for all $B_1, B_2 \in \mathcal{B}$, if $x \in B_1 \cap B_2$, then $\exists B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$

A basis induces a topology on a non-empty set X . This topology is called the topology generated by the basis.

Definition 6. Let X be a non-empty set and let \mathcal{B} be a basis on X . Define τ to be the collection of all sets $U \subset X$ such that given any $x \in U$ we can find a basis element $B \in \mathcal{B}$ such that $x \in B \subset U$.

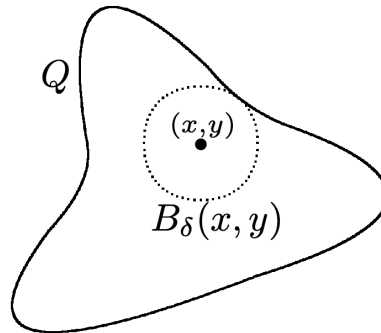


Figure 2.2: In \mathbb{R}^2 (cr. Crossley, Essential Topology pg. 19)

Definition 7. Let (X, τ) be a topological space. A set \mathcal{B} of subsets of X is called a basis for τ if \mathcal{B} is a basis and the topology generated by \mathcal{B} is equal to τ .

Example 5. Consider $\mathcal{B} = \{(a, b) \mid a, b \in \mathbb{R}\}$. We can check that \mathcal{B} is a basis. The topology generated by \mathcal{B} is called the standard topology.

Example 6. Let X and Y be topological spaces. We define the product topology on $X \times Y$ as the topology whose basis is the collection of all $Z_1 \times Z_2$ where $Z_1 \in \tau_X$ and $Z_2 \in \tau_Y$.

Example 7. Let (\mathbb{R}^n, τ_X) . We define the standard topology on \mathbb{R}^n as the topology whose basis is the collection of all $B_\epsilon(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^n \mid d_E(\mathbf{x}, \mathbf{y}) < \epsilon\}$. Here the d_E function is Euclidean distance, defined in metrics.

Definition 8. Let (X, τ) be a topological space. We say a set $U \subset X$ is open in X if and only if $U \in \tau$.

Example 8. The interval $(1, 2] = [0, 2] \cap (1, 3)$ is an open subset of $[0, 2]$ by the subspace topology.

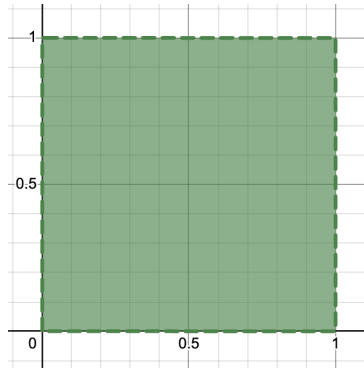
Definition 9. Let X be a topological space and $V \subset X$. We say V is closed in X if and only if $X \setminus V$ is open.

Example 9. Let X be a topological space. Then \emptyset is closed, since $X \setminus \emptyset = X$, which is open in itself. Similarly, X is closed, since $X \setminus X = \emptyset$ is open.

Example 10. The set $[0, \infty) = \mathbb{R} \setminus (-\infty, 0)$ is closed in \mathbb{R} .

Definition 10. Let X be a topological space and $J \subset X$. We say $x \in J$ is an interior point of J if there exists open $U \subset X$ such that $x \in U \subset J$.

Definition 11. Let X be a topological space and $J \subset X$. We define the interior of J as $\text{int}(J) := \{x \in J \mid x \text{ is an interior point of } J\}$.

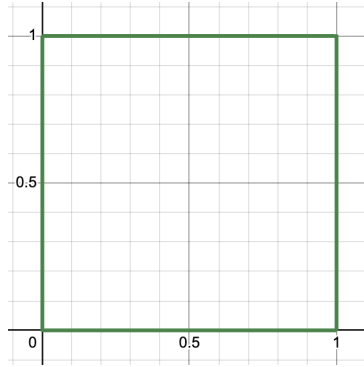


Example 11. The interior of $[0, 1] \times [0, 1]$ is $(0, 1) \times (0, 1)$.

Definition 12. Let X be a topological space and $x \in X$. Let $J \subset X$ such that $x \in J$. We say x is a boundary point of J if for any open $U \subset X$ such that $x \in U$, $U \cap J \neq \emptyset$ and $U \cap (X \setminus J) \neq \emptyset$.

Definition 13. Let X be a topological space and $J \subset X$. We define the boundary of J as $\partial(J) := \{x \in J \mid x \text{ is a boundary point of } J\}$.

Example 12. The boundary of $[0, 1] \times [0, 1]$ is the union of the four line segments of the square.



Definition 14. Let M be a non-empty set. We define a metric d on M , as $d : M \times M \rightarrow \mathbb{R}$ satisfying the following properties for any $x, y, z \in \mathbb{R}$:

1. $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$

In this thesis we will only explicitly consider two metrics.

Example 13. Euclidean distance: For any $\mathbf{x} = (x_1, x_2, \dots, x_n), \mathbf{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ we define $d_E(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots + (x_n - y_n)^2}$.

Example 14. River metric: For any $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, we define

$$d^*((x_1, y_1), (x_2, y_2)) = \begin{cases} |y_1 - y_2| & \text{if } x_1 = x_2 \\ |y_1| + |y_2| + |x_1 - x_2| & \text{otherwise} \end{cases}$$

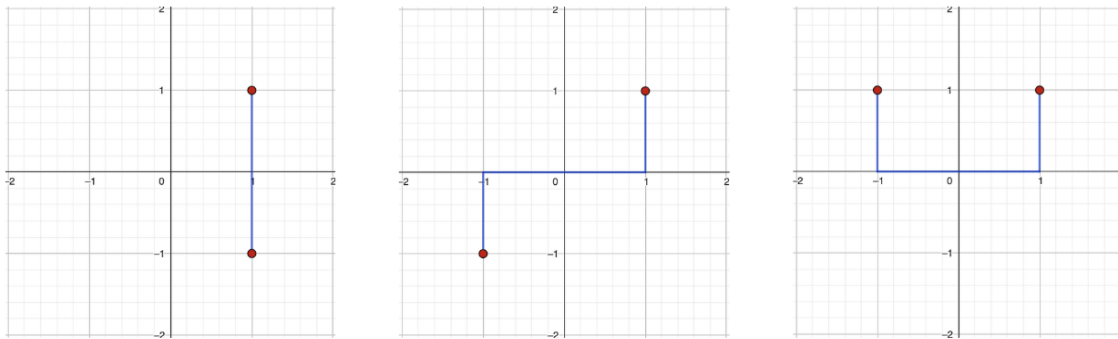


Figure 2.3: The river metric

We can replace the Euclidean metric in \mathbb{R}^2 with this metric. To understand open balls in the river metric, I have created this file in [Geogebra](#). If you would like to read more about the river metric and understand it in detail, refer [here](#).

Definition 15. We define a metric space as the pair (M, d) where d is a metric on a non-empty set M .

Note: for ease of notation, while writing we replace (M, d) with M as a metric space.

Definition 16. Let X be a topological space. We define a sequence (x_n) as a function $f : \mathbb{N} \rightarrow X$ where $x_n = f(n)$.

Definition 17. Let X be a topological space and (x_n) be a sequence. We define a subsequence (x_{n_k}) as a composition of a f with a strictly increasing map $g : \mathbb{N} \rightarrow \mathbb{N}$.

Definition 18. Let M be a metric space. We say $S \subset X$ is bounded if and only if there exists $x \in M$ and $r > 0$ such that $\forall s \in S$, we have $d(x, s) < r$.

Definition 19. Let X be a topological space and (x_n) be a sequence contained in X . We say (x_n) converges to $x \in X$ if and only if for any open $U \subset X$ such that $x \in U$, $\exists N \in \mathbb{N}$ such that $\forall n > N$, $x_n \in U$.

Definition 20. Let \leq be a binary relation of a set P i.e. $\leq \subset P \times P$. We say \leq is a preorder if it is reflexive ($a \leq a$) and transitive ($a \leq b, b \leq c \implies a \leq c$).

Definition 21. Let X be a topological space. We define (X, \leq) to be a directed set where \leq is a preorder on X and every finite subset of X has an upper bound.

For ease of notation, while writing we replace (A, \leq) with A as a directed set. In the thesis we will only use it for the next definition.

Definition 22. Let X be a topological space and (A, \leq) be a directed set. We define a net (x_t) as a function $g : A \rightarrow X$ where $x_t = g(t)$.

A net can be thought of as a generalisation of sequences, used for less ‘nice’ topological spaces, as sequences fail to characterise and capture properties, such as for sets with an uncountable

basis. We will use nets as a replacement for sequences when working with more general spaces.

Definition 23. Let $f : X \rightarrow Y$ be a function. We say f is continuous if and only if for any open $V \subset Y$, the pre-image of V under f is an open subset of X .

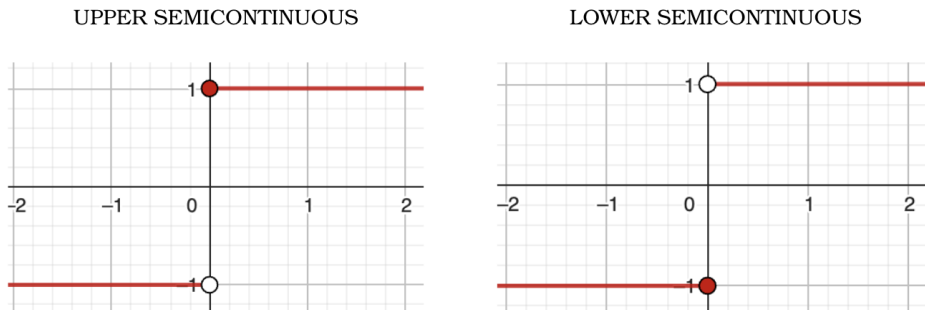
Continuous functions are the heart of this thesis and we will see many examples of continuous and non-continuous functions while reading.

Definition 24. Let $f : X \rightarrow Y$ be a function and $x \in X$. We say f is continuous at x if and only if for any open $V \subset Y$ such that $f(x) \in V$, there exists open $U \subset X$ such that $x \in U$ and $f(U) \subset V$.

Counterexample 1. The function $f : [0, 1] \rightarrow \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ for non-zero x and 0 at 0 is not continuous at 0.

Definition 25. Let X be a topological space and $f : X \rightarrow \mathbb{R}$. Let $x_0 \in X$. We say f is upper-semicontinuous at x_0 if for any $y > f(x_0)$, we can find open $U \subset X$ containing x_0 such that $f(x) < y$ for all $x \in U$.

Definition 26. Let X be a topological space and $f : X \rightarrow \mathbb{R}$. Let $x_0 \in X$. We say f is lower-semicontinuous at x_0 if for any $y < f(x_0)$, we can find open $U \subset X$ containing x_0 such that $f(x) > y$ for all $x \in U$.



Example 15. The Dirichlet function defined as $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$ is upper-semicontinuous at rational points, lower-semicontinuous at irrational points, and not continuous anywhere.

Definition 27. Let X and Y be topological spaces. We say X is homeomorphic to Y if there exists a continuous bijective function $f : X \rightarrow Y$ whose inverse is also continuous. The function f is called a homeomorphism.

Example 16. The topological spaces $(X, \mathcal{P}(X))$ and $(Y, \mathcal{P}(Y))$ are homeomorphic iff there is a bijection between X and Y . In particular, the topological spaces $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ and $(\mathbb{Z}, \mathcal{P}(\mathbb{Z}))$ are homeomorphic to each other.

Example 17. Note that $S^1 \setminus \{(0, 1)\}$ (the unit circle minus a point) is a subset of \mathbb{R}^2 and thus it inherits the subspace topology from \mathbb{R}^2 equipped with the standard topology. Then, $S^1 \setminus \{0\}$ and \mathbb{R} are homeomorphic to each other.



Figure 2.4: Homeomorphic spaces (cr. Henry Segerman)

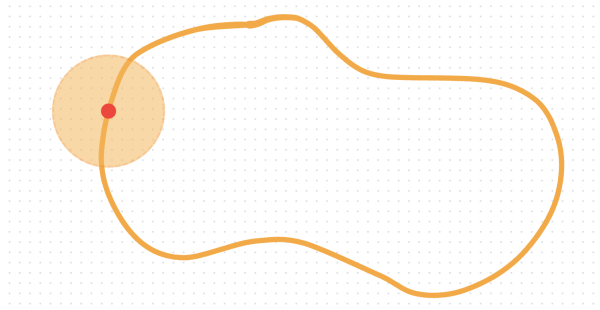
Counterexample 2. The topological spaces (a, b) and $[a, b]$ (equipped with the subspace topology from \mathbb{R}) are not homeomorphic to each other.

Counterexample 3. The topological spaces \mathbb{R} and \mathbb{R}^2 (equipped with standard topology) are not homeomorphic to each other.

Definition 28. Let X be a topological space and $E \subset X$. We call $x \in X$ a limit point of E if for any open $U \subset X$ such that $x \in U$, $U \cap (E \setminus \{x\}) \neq \emptyset$.

Definition 29. Let X be a topological space. We say X is connected if and only if there do not exist two disjoint non-empty open sets $U, V \subset X$ such that $U \cup V = X$.

Example 18. The set $(0, \infty)$ is a connected subset of \mathbb{R} .

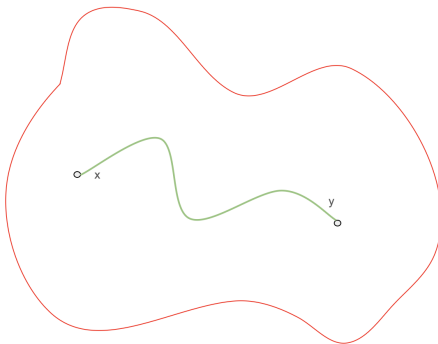


Example 19. *The set $\mathbb{R}^2 \setminus \{0\}$ is connected in \mathbb{R}^2 .*

Example 20. *The graph of a continuous function $f : X \rightarrow Y$ is always connected in $X \times Y$.*

Counterexample 4. *The set $\mathbb{R} \setminus \{0\}$ is not connected in \mathbb{R} .*

Definition 30. *Let X be a topological space. We say X is path-connected if and only if for any two points $x, y \in X$, we can find a continuous ‘path’ between them: $\lambda : [0, 1] \rightarrow X$ such that $\lambda(0) = x$ and $\lambda(1) = y$.*



Path-connectedness is stronger than connectedness, but they are not equivalent ideas. The graph of the topologist’s sine curve is an example of a set that is connected, but not path-connected.

Definition 31. *Let X be a topological space. We say \mathcal{U} is an open cover of X if and only if $\bigcup_{U \in \mathcal{U}} U = X$ and every $U \in \mathcal{U}$ is open in X .*

Definition 32. *Let X be a topological space. Suppose \mathcal{U} is an open cover of X . We say X is compact if and only if we can find an open cover \mathcal{U}' of X such that $\mathcal{U}' = \{U_1, U_2, \dots, U_n\} \subset \mathcal{U}$, called a ‘finite refinement’.*

An alternate but powerful definition for compactness lies in subsequences. We will use this definition as well (keep in mind that we must work in a ‘nicer’ space for sequential compactness).

Definition 33. *Let M be a metric space. We say M is compact if and only if every sequence in M has a convergent subsequence.*

Example 21. *The set $[a, b]$ is a compact subset of \mathbb{R} .*

Example 22. *The unit circle S^1 is a compact subset of \mathbb{R}^2 .*

Counterexample 5. *The set (a, b) is not a compact subset of \mathbb{R} .*

Counterexample 6. *The set \mathbb{Q} is not a compact subset of \mathbb{R} .*

Definition 34. *Let X be a topological space and $x \in X$. We say X is locally connected at x if and only if given open $U \subset X$ such that $x \in U$, there exists connected open $K \subset X$ such that $x \in K \subset U$.*

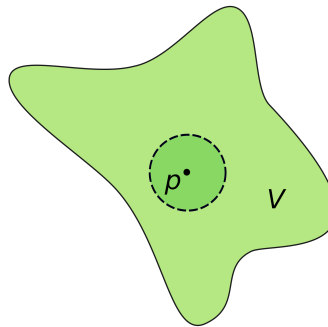


Figure 2.5: cr. Oleg Alexandrov

If this holds at every point, X is a locally connected space.

Example 23. *The space $(0, 1) \cup (2, 3)$ is locally connected.*

Example 24. *The space $\mathbb{R} \setminus \{0\}$ is locally connected.*

Definition 35. *Let X be a topological space and $x \in X$. We say X is locally compact at x if and only if there exists open $U \subset X$ such that $x \in U$ and \bar{U} is compact.*

If this holds at every point, X is a locally compact space.

Example 25. For any $n \in \mathbb{N}$, \mathbb{R}^n is a locally compact space.

Definition 36. Let X, Y be topological spaces and $f : X \rightarrow Y$. We say f is a Darboux function if for any connected $E \subset X$, $f(E)$ is a connected subset of Y .

Definition 37. Let X, Y be topological spaces and $f : X \rightarrow Y$. We say f is a connectivity function if for any connected $E \subset X$, $Gr(f|_E)$ is a connected subset of $X \times Y$.

2.2 Lemmas

Lemma 1. Let $b_n = (a_{1n}, a_{2n}, \dots, a_{mn})$ be a sequence. Then (b_n) converges to (a_1, a_2, \dots, a_m) if and only if $(a_{in}) \rightarrow a_i, \forall i \in \{1, 2, \dots, m\}$. In other words, the sequence converges iff each component sequence converges to its limit component.

Proof. Outline: Given $\epsilon > 0$, we make each $|a_{n_1} - a_i|$ smaller than $\frac{\epsilon}{m}$, where $n = \max\{N_1, \dots, N_m\}$. □

Lemma 2. Let X be a topological space. If $A, B \subset X$ are disjoint and open, $\overline{A} \cap B = A \cap \overline{B} = \emptyset$.

Proof. Let $x \in B$. Since B is open, we can find open $U \subset X$ such that $x \in U \subset B$. Since $A \cap B = \emptyset$, $A \cap U = \emptyset$. Since U was arbitrary, x is not a limit point of A , i.e. $x \notin \overline{A}$. Using a parallel argument, $A \cap \overline{B} = \emptyset$. □

Lemma 3. Let $f : \mathbb{R} \rightarrow Y$ have a connected graph. Then f is a Darboux function.

Proof. Let $E \subset \mathbb{R}$ be connected. We want to show that $f(E)$ is connected. Suppose not. Then there exist disjoint non-empty open $U, V \subset f(E)$ such that $f(E) = U \cup V$. Consider $Gr(f|_E)$. Connectivity and a connected graph are equivalent for functions from \mathbb{R} , so this is also a connected set. However, $E_1 = Gr(f|_E) \cap (E \times U)$ and $E_2 = Gr(f|_E) \cap (E \times V)$ form a separation of this set. This is a contradiction so $f(E)$ is connected and f is Darboux. □

Lemma 4. If $f : X \rightarrow \mathbb{R}$, then $int(Gr(f)) = \emptyset$.

Proof. Let U be an open subset of $X \times \mathbb{R}$. Suppose $U \subset Gr(f)$. Then, since U is open there exist two open subsets $E \subset X$ and $F \subset \mathbb{R}$ such that $(x, f(x)) \in E \times F \subset Gr(f)$. Since F is open, we can find $\epsilon > 0$ such that $f(x) \in (f(x) - \epsilon, f(x) + \epsilon) \subset F$. Then, $(x, f(x) + \epsilon/2) \in E \times F$, and clearly this point is not in the graph. However, $E \times F \subset Gr(f)$, which is a contradiction. By the definition of the interior we have that $\text{int}(Gr(f)) = \emptyset$. \square

Lemma 5. *Let X and Y be topological spaces and $x \in X$. The set $\{x\} \times Y$ is homeomorphic to Y .*

Proof. Define $f : \{x\} \times Y \rightarrow Y$ as $(x, y) \mapsto y$.

Let $(x, y_1), (x, y_2) \in \{x\} \times Y$ such that $f((x, y_1)) = f((x, y_2))$. Then $y_1 = y_2$ so $(x, y_1) = (x, y_2)$. Let $y \in Y$. Then $\exists(x, y) \in \{x\} \times Y$ such that $f(x, y) = y$. The function is both injective and surjective, so it is a bijective function.

We will now show that f and $f^{-1} : Y \rightarrow \{x\} \times Y$ defined as $f^{-1}(y) = (x, y)$, are continuous functions. Proving the first, $V \subset Y$ be open. By the product topology, $\{x\} \times V \subset \{x\} \times Y$ is open, and this is the pre-image of our open set Y . Since V was arbitrary, f is continuous. Let $U \subset \{x\} \times Y$ be open. By the product topology, we can find open $V \subset Y$ such that $U = \{x\} \times V$. Then $f(\{x\} \times V) = V$ is open in Y . Since U was arbitrary, f^{-1} is also continuous. \square

Homeomorphisms preserve topological properties. By Proposition 5.16 in Essential Topology (Crossley 2006), if two spaces are homeomorphic:

- the image of a closed set under a homeomorphism is closed
- the image of a connected set under a homeomorphism is connected
- the image of a compact set under a homeomorphism is compact

In other words, the function preserves topological properties.

Lemma 6. *Let $f : X \rightarrow Y$ be a function. Then f is continuous if and only if for any closed $V \subset Y$, $f^{-1}(V)$ is a closed subset of X .*

Proof. Suppose for any closed $V \subset Y$, the pre-image of V under f is a closed subset of X . Let Z be an open subset of Y . Then $Y \setminus Z$ is closed. Let $U = f^{-1}(Z)$. By our supposition, $f^{-1}(Y \setminus Z)$ is closed. Since f is a well-defined function, $f^{-1}(Y \setminus Z) = f^{-1}(Y) \setminus f^{-1}(Z) = X \setminus U$ is closed. Therefore, U is open. Since Z was arbitrary, f is a continuous function.

Suppose f is continuous. Let V be a closed subset of Y . Then $Y \setminus V$ is an open set. Let $U = f^{-1}(V)$. Since f is continuous $f^{-1}(Y \setminus V)$ is open. Since f is a well-defined function, $f^{-1}(Y \setminus V) = f^{-1}(Y) \setminus f^{-1}(V) = X \setminus U$. So U is closed. Therefore, every closed set has a closed pre-image. \square

Lemma 7. *Let X be a topological space and $f : X \rightarrow \mathbb{R}$. Then f is upper semi-continuous if and only if the set $A = \{(x, y) : f(x) < y\}$ is open in $X \times \mathbb{R}$. Similarly, f is lower semi-continuous if and only if the set $B = \{(x, y) : f(x) > y\}$ is open in $X \times \mathbb{R}$. Consequently, f is continuous if and only if the sets A and B are open.*

Proof. We typically talk about semi-continuity for functions $f : X \rightarrow \mathbb{R}$ (which is what we're working with!).

We will first show that if f is upper semi-continuous, then A is open. Let $(x_0, y_0) \in A$. From our definition of A it follows that $f(x_0) < y_0$. Then, there exists $\epsilon > 0$ such that $f(x_0) < y_0 - \epsilon$ (for instance, $\epsilon = \frac{y_0 - f(x_0)}{2}$). By the upper semi-continuity of f , $\exists U \subset_{\text{open}} X$ such that $x_0 \in U$ and $f(x) < y_0 - \epsilon, \forall x \in U$. Define $V = (y_0 - \epsilon, \infty)$. Clearly, $(x_0, y_0) \in U \times V$. Let $(x, y) \in U \times V$. Since $x \in U$, $f(x) < y_0 - \epsilon$ and since $y \in V$, $y_0 - \epsilon < y$. By transitivity, $f(x) < y$, i.e. $(x, y) \in A$. Thus, $U \times V \subset A$. As (x_0, y_0) was arbitrary and we were able to find $U \subset_{\text{open}} X$ and $V \subset_{\text{open}} \mathbb{R}$ such that $(x_0, y_0) \in U \times V \subset A$, A is open by the product topology.

Now we will show that if A is open, then the function is upper semi-continuous. Let $x_0 \in X$. Suppose $y_0 \in \mathbb{R}$ such that $f(x_0) < y_0$. Then, $(x_0, y_0) \in A$. Since A is an open subset of $X \times \mathbb{R}$, there exists $U \subset_{\text{open}} X$ and $V \subset_{\text{open}} \mathbb{R}$ such that $x_0 \in U$, $y_0 \in V$ and $U \times V \subset A$. For any $x \in U$, $(x, y_0) \in U \times V$. Then $f(x) < y_0$ (since $U \times V \subset A$). We could find an open set U such that $\forall x \in U$, $f(x) < y_0$. As y_0 was arbitrary, f is upper semicontinuous at x_0 . As x_0 was

arbitrary, f is upper semicontinuous.

Analogously, f is lower semi-continuous if and only if the set $B = \{(x, y) : f(x) > y\}$ is open in $X \times \mathbb{R}$.

The consequence follows since any function is continuous if and only if it is upper and lower semi-continuous. As shown above, semi-continuity of f is equivalent to the sets A and B being open subsets of $X \times \mathbb{R}$. \square

The symbol $[c]$ denotes uncountable cardinality.

Lemma 8. *If $E \subset (0, \infty)$ has size $[c]$, then there is a surjection $f : E \rightarrow (0, \infty)$ such that $f(x)/x \notin \mathbb{Q}$ for every $x \in E$.*

Proof. Suppose E is uncountable and $\exists y \in (0, \infty)$ such that $\forall x \in E, y/x \in \mathbb{Q}$. Then E would have at most the cardinality of \mathbb{Q} (a countable set) since we can construct a bijective function $h : E \rightarrow \mathbb{Q}$ defined as $h(x) = y/x$. This contradicts our assumption that E is uncountable. Thus, if $E \subset (0, \infty)$ is uncountable, then for every $y \in (0, \infty)$ there exists an $x \in E$ with $y/x \notin \mathbb{Q}$. \square

Lemma 9. *If $f : X \rightarrow Y$ is a connectivity function, then f is Darboux. Equivalently, if f is not Darboux, then f is not a connectivity function. In other words, if there exists a connected $E \subset X$ such that $f(E)$ is disconnected, then $Gr(f|_E)$ is not connected.*

Proof. We are given that $f(E)$ is not connected i.e. there exist non-empty open subsets of Y , U and V , such that $f(E) = U \cup V$ and $U \cap V = \phi$.

Now, as U and V are non-empty, there exists $y_U \in U$ and $y_V \in V$. Since $U \subset f(E)$, there exists $x_U \in E$ such that $f(x_U) = y_U$. Notice that as $x_U \in E \subset X$ and $y_U \in U$, $(x_U, y_U) \in X \times U$. Additionally, as $x_U \in E$, $(x_U, y_U) = (x_U, f(x_U)) \in Gr(f|_E)$. Therefore, $(x_U, y_U) \in (X \times U) \cap Gr(f|_E)$. Similarly, there exists $x_V \in E$ such that $f(x_V) = y_V$ and $(x_V, y_V) \in (X \times V) \cap Gr(f|_E)$.

Further, $(X \times U) \cap Gr(f|_E)$ and $(X \times V) \cap Gr(f|_E)$ are open subsets in the subspace topology, since $X \times U$ and $X \times V$ are open subsets of $X \times Y$ in the product space topology, and $Gr(f|_E)$

is a subset of $X \times Y$. Therefore $\mathcal{U} = (X \times U) \cap Gr(f|_E)$ and $\mathcal{V} = (X \times V) \cap Gr(f|_E)$, are non-empty open subsets of $Gr(f|_E)$.

Now we can prove that the sets are disjoint,

$$\begin{aligned}
\mathcal{U} \cap \mathcal{V} &= ((X \times U) \cap Gr(f|_E)) \cap ((X \times V) \cap Gr(f|_E)) \\
&= ((X \times U) \cap (X \times V)) \cap Gr(f|_E) \\
&= (X \times (U \cap V)) \cap Gr(f|_E) \\
&= \phi && \text{(since } U \cap V = \phi)
\end{aligned}$$

and their union is the whole set.

$$\begin{aligned}
\mathcal{U} \cup \mathcal{V} &= ((X \times U) \cap Gr(f|_E)) \cup ((X \times V) \cap Gr(f|_E)) \\
&= ((X \times U) \cup (X \times V)) \cap Gr(f|_E) && \text{(by the Set Distributive Law)} \\
&= (X \times (U \cup V)) \cap Gr(f|_E) && \text{(C.P. is distributive over unions)} \\
&= (X \times f(E)) \cap Gr(f|_E) \\
&= Gr(f|_E) && \text{(since } Gr(f|_E) \subset E \times f(E) \subset X \times f(E))
\end{aligned}$$

Therefore, $Gr(f|_E)$ is not a connected set, and this is not a connectivity function. □

Lemma 10. *Let X be a compact space. If $U \subset X$ is closed, then U is compact.*

Proof. Let \mathcal{U} be an open cover of U . Then, since U is closed, $X \setminus U$ is open and $\mathcal{U} \cup X \setminus U$ is an open cover of X . Since X is compact, we can find a finite refinement of this cover, say $U_1, U_2, \dots, U_n, X \setminus U$. Then, U_1, U_2, \dots, U_n is a finite subcover of \mathcal{U} . Since \mathcal{U} was an arbitrary open cover of U that admitted finite refinement, U is compact. □

In this thesis, we will use the idea that the continuous image of a connected or compact set is connected or compact (generalised versions of the intermediate and extreme value theorems).

Chapter 3

Connectedness and Related Ideas

The graph of the topologist's sine curve locally behaves oddly around zero. We can actually prove that ensuring the graph is locally connected is equivalent to continuity, at least for functions from \mathbb{R} (regardless of codomain). We will also compare connectivity functions and Darboux functions to understand how we can use them for continuity. Finally, we will see how continuity ensures a connected graph disconnects the space it belongs to (think about lifting the graph out of the page).

3.1 Locally Connected Graphs

We will first try to characterise continuous functions by locally connected graphs.

3.1.1 Connected Graphs

Fact 3.1.1. *Let Y be an arbitrary topological space and $f : \mathbb{R} \rightarrow Y$ have a connected graph. Then f is a connectivity function.*

Fact 3.1.1 holds in large part because the domain of the function is \mathbb{R} . Connectedness is otherwise weak with regards to characterising connectivity, as demonstrated in the example below.

Example 3.1.1. *The function $\psi : S^1 \rightarrow \mathbb{R}$ defined by $(\sin(t), \cos(t)) = \frac{1}{t}$ for $t \in (0, 2\pi]$ is not*

a connectivity function, although it has a connected graph.

We show in Claim 3 of Example 5.1.1 that ψ has a connected graph.

To prove that it is not a connectivity function we will use Lemma 9 – every connectivity function is Darboux. For proving the reverse implication of this false, see Example 3.1.3!

Proof. It is sufficient to prove that the ψ function is not Darboux. Let us take the connected set $E_\epsilon = \{(\cos(t), \sin(t)) \mid t \in (0, \epsilon) \cup (2\pi - \epsilon, 2\pi]\}$. This is an arc of the unit circle S^1 around $(0, 1)$.

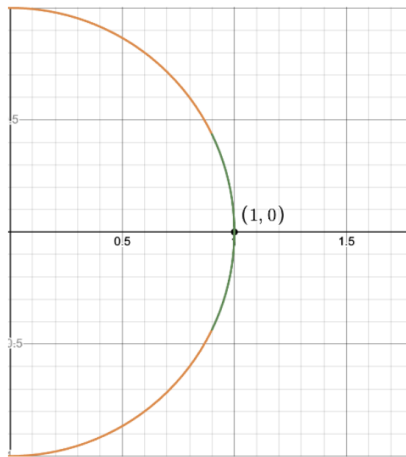


Figure 3.1: A closer look at our domain.

Now,

$$\begin{aligned} \psi(E_\epsilon) &= \left\{ \frac{1}{t} \mid t \in (0, \epsilon) \cup (2\pi - \epsilon, 2\pi] \right\} \\ &= \left[\frac{1}{2\pi}, \frac{1}{2\pi - \epsilon} \right) \cup \left(\frac{1}{\epsilon}, \infty \right) \end{aligned}$$

What ϵ do we choose? Suppose we take $\epsilon < \pi$. Then,

$$\begin{aligned} \epsilon < \pi &\implies 2\epsilon < 2\pi \\ &\implies 2\epsilon - \epsilon = \epsilon < 2\pi - \epsilon \\ &\implies \frac{1}{2\pi - \epsilon} < \frac{1}{\epsilon} \text{ (i.e. disjoint intervals)} \end{aligned}$$

So, let $\epsilon = \frac{\pi}{2}$ (any value in $(0, \pi)$ works). Then, $\psi(E_{\frac{\pi}{2}}) = [\frac{1}{2\pi}, \frac{2}{3\pi}) \cup (\frac{2}{\pi}, \infty)$. These intervals are non-empty by definition and disjoint since $\frac{2}{3\pi} < \frac{6}{3\pi} = \frac{2}{\pi}$.

Now, $[\frac{1}{2\pi}, \frac{2}{3\pi})$ and $(\frac{2}{\pi}, \infty)$ are open subsets of $\psi(E_{\frac{\pi}{2}})$ by the subspace topology: $[\frac{1}{2\pi}, \frac{2}{3\pi}) = \psi(E_{\frac{\pi}{2}}) \cap (0, \frac{2}{3\pi})$ and $(\frac{2}{\pi}, \infty) = \psi(E_{\frac{\pi}{2}}) \cap (\frac{2}{\pi}, \infty)$. These two sets form a separation of $\psi(E_{\frac{\pi}{2}})$, so it is *not* a connected set.

Since $E_{\frac{\pi}{2}}$ is a connected set, we have proven that ψ is not Darboux. By Lemma 9, we have that $Gr(f|_{E_{\frac{\pi}{2}}})$ is not connected. The existence of this $E_{\frac{\pi}{2}}$ gives us that ψ is not a connectivity function. \square

Theorem 3.1.1. *Let Y be an arbitrary topological space. Let $F : \mathbb{R} \rightarrow Y$ be a connectivity function. Suppose that $Gr(F)$ is locally connected at $(x, F(x))$. Then, F is continuous at x .*

Proof. Let $V \subset Y$ be open and contain $F(x)$. Then, $(x, F(x)) \in Gr(F) \cap (\mathbb{R} \times V)$, open in $Gr(F)$. There must exist a connected set, say K , s.t. $(x, F(x)) \in K \subset Gr(F) \cap (\mathbb{R} \times V)$, since $Gr(F)$ is locally connected a point.

Suppose that there exist *no* elements greater than x in $dom(K)$. So, $K \subset (-\infty, x] \times Y$. Then we can show that $\{(x, F(x))\}$ is both a closed and open subset of $Gr(F|_{[x, \infty)})$.

Let us look at the set $W_1 = Gr(F|_{[x, \infty)}) \cap K$. It is an open subset of $Gr(F|_{[x, \infty)})$ by the subspace topology, since K is an open subset of $Gr(F)$. Since $Gr(F|_{[x, \infty)}) \subset [x, \infty) \times Y$ and $K \subset (-\infty, x] \times Y$ (by our initial assumption), we have that W_1 is a subset of their intersection, $\{x\} \times Y$. We know that $W_1 \subset Gr(F|_{[x, \infty)})$, so together $\{(x, F(x))\} = W_1$, which is open.

Notice that $(-\infty, x] \times Y$ is a closed subset of $\mathbb{R} \times Y$ since $(-\infty, x] = \mathbb{R} \setminus (x, \infty)$ is closed in \mathbb{R} and Y is closed in itself. Let us look at the set $W_2 = Gr(F|_{[x, \infty)}) \cap ((-\infty, x] \times Y)$. It is a closed subset of $Gr(F|_{[x, \infty)})$ by the subspace topology. Additionally, $W_2 \subset ((-\infty, x] \times Y) \cap ([x, \infty) \times Y) = \{x\} \times Y$. We know that $W_2 \subset Gr(F|_{[x, \infty)})$, so together $\{(x, F(x))\} = W_2$, which is closed.

Thus, $\{(x, F(x))\}$ is clopen, so its complement, $Gr(F|_{[x, \infty)}) \setminus \{(x, F(x))\}$, is also clopen. These sets are disjoint, non-empty open subsets of $Gr(F|_{[x, \infty)})$ that cover it. However, we were given

that F was a connectivity function, so $Gr(F|_{[x,\infty)})$ must be a connected set.

This is a contradiction, so there exists $b \in dom(K)$ such that $b > x$. Using a parallel argument (suppose $K \subset [x, \infty) \times Y$), we can prove that there exists $a \in dom(K)$ such that $a < x$.

Let $U = (a, b)$. Then, $x \in U \subset dom(K)$, since the $dom(K)$ is a connected set. All elements of U are in the projection of K onto \mathbb{R} , so by how we've defined $dom(K)$, for every $u \in U$, $F(u) \in V$ (since $K \subset Gr(F)$). Thus, $F(U) \subset V$.

Given an open $V \subset Y$ s.t. $F(x) \in V$, we found an open subset U of \mathbb{R} ((a, b) is an open interval) s.t. $x \in U$, and whose image under F was contained in V . Thus, the function is continuous at x . □

Would it still hold true in the other direction? Even for a less restricted X ?

Theorem 3.1.2. *Let X be a locally connected space. Let Y be an arbitrary topological space. Let $F : X \rightarrow Y$ be a connectivity function, continuous at x . Then $Gr(F)$ is locally connected at $(x, F(x))$.*

Proof. Let V' be an open subset of $Gr(F)$. By the subspace topology, there exists an open subset of $X \times Y$, say V , containing $(x, F(x))$ such that $V' = Gr(F) \cap V$. Since V is open, there exists a basis element, say E , such that $(x, F(x)) \in E \subset V$. For the product topology, basis elements are of the form $B \times U$ where B and U are open subsets of X and Y , respectively. Thus, we say $E = B \times U$.

Since F is continuous at x , we can find an open $G_1 \subset X$ s.t. $x \in G_1$ and $F(G_1) \subset U$. Let $G_2 = G_1 \cap B$, which is open, be completely contained in B and also satisfy $x \in G_2$ and $F(G_2) \subset U$.

Additionally, we have that X is locally connected, so by definition, we can find an open connected subset of G_2 , say G , which contains x . Since F is a connectivity function and G is connected, the set $Gr(F|_G)$ is connected by definition.

Since $F(G) \subset U$,

$$Gr(F|_G) = Gr(F) \cap (G \times U)$$

This is an open subset of $Gr(F)$ since $G \times U$ is an open subset of $X \times Y$ by the product topology and $Gr(F) \subset X \times Y$.

Since $G \subset B$, $Gr(F|_G) = Gr(F) \cap (G \times U) \subset B \times U = E \subset V$. Additionally, $Gr(F|_G) \subset Gr(F)$, so $Gr(F|_G) \subset Gr(F) \cap V = V'$.

The set $Gr(F|_G)$

- contains $(x, F(x))$
- is an open subset of $Gr(F)$
- is a connected subset of $Gr(F)$
- is contained in V'

By definition, $Gr(F)$ is locally connected at $(x, F(x))$. □

We will combine the ideas of the previous two theorems to obtain a powerful result.

Theorem 3.1.3. *Let Y be an arbitrary topological space. Let $F : \mathbb{R} \rightarrow Y$ be a function with a connected graph. Then F is continuous at x if and only if $Gr(F)$ is locally connected at $(x, F(x))$.*

Proof. From Fact 3.1.1, we have that since Y is an arbitrary topological space and $F : \mathbb{R} \rightarrow Y$ has a connected graph, F is a connectivity function.

Suppose that this graph is also locally connected at $(x, F(x))$. Theorem 3.1.1 gives us that, F is continuous at x .

Instead, suppose that the graph is continuous at x . Since \mathbb{R} is a locally connected space, Theorem 3.1.2 yields $Gr(F)$ is locally connected at $(x, F(x))$. □

So, a locally connected graph was necessary for the topologist's sine curve! This is one way it failed to ensure continuity. Note that the domain \mathbb{R} we've used in Theorem 3.1.3 is as nice as any space can be. However, moving to other domains can cause issues, as illustrated by the following example.

Example 3.1.2. *There is a function $F : X \rightarrow [0, 1]$ where X is a compact convex subset of the Euclidean plane, thus locally connected. $Gr(F|_G)$ is path-connected for every open connected set $G \subset X$, and the graph of F is locally path-connected. Yet, F is not continuous.*

Proof. Let $X = \{(x, y) : (0 \leq x \leq 1) \wedge (0 \leq y \leq x)\}$. Let $F : X \rightarrow [0, 1]$ be defined as:

$$F(x, y) = \begin{cases} \frac{2xy}{x^2+y^2} & \text{if } y > 0 \\ 0 & \text{if } y = 0 \end{cases}$$

1. We will prove that F is discontinuous.

For every $a \in [0, 1]$ let $K_a = \{(x, ax) \mid x \in [0, 1]\}$.

Then,

$$\begin{aligned} F(K_a) &= \{F(x, ax) \mid x \in [0, 1]\} \\ &= \{2x(ax)/(x^2 + (ax)^2) \mid x \in (0, 1]\} \cup \{0\} \\ &= \{x^2(2a)/x^2(1 + a^2) \mid x \in (0, 1]\} \cup \{0\} \\ &= \{2a/(1 + a^2)\} \cup \{0\} \\ &= \{0, 2a/(1 + a^2)\} \end{aligned}$$

Although K_a (a line) is a connected set, its image (two distinct points) is not connected. Thus, F cannot be a continuous function, since every continuous function is Darboux, i.e. the continuous image of a connected set is connected. □

[To Note: As F is a rational function whose denominator $x^2 + y^2 = 0$ iff $(x, y) = (0, 0)$, it is continuous at all points except $(0, 0)$. This is the point of discontinuity. On the restricted

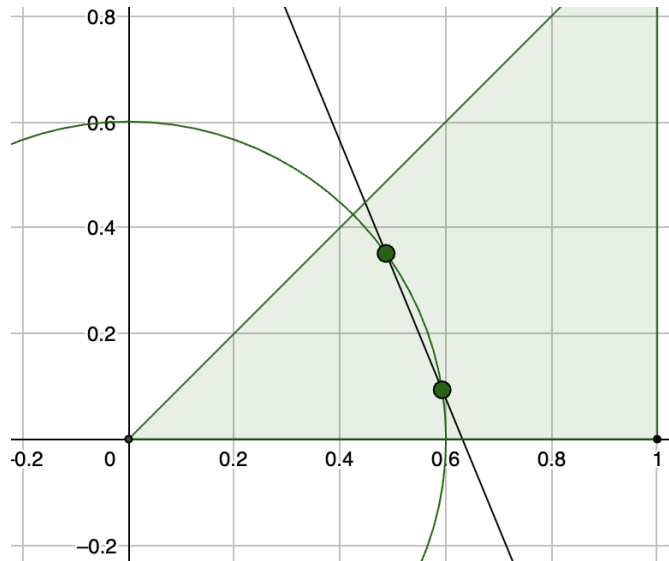
domain $X \setminus \{(0, 0)\}$, we've defined the function to be $F(x, y) = 2xy/(x^2 + y^2)$. This is a product of continuous functions, so it is continuous.]

2. We can prove however, that if G is a connected open subset of X , then $Gr(F|_G)$ is connected.

Lemma. $G \setminus \{(0, 0)\}$ is path-connected and hence connected.

Proof. Let x and y be two arbitrary points in $G \setminus \{(0, 0)\}$. Then, as $x, y \in G$ and G is path connected (because it is open and connected), there exists a continuous curve $\gamma : [0, 1] \rightarrow G$ such that $\gamma(0) = x$ and $\gamma(1) = y$. If $\gamma(t) \neq (0, 0)$ for any t , then γ is a path in $G \setminus \{(0, 0)\}$. Otherwise, as G is open, there exists an ε such that $B((0, 0), \varepsilon) \cap X \subset G$. Define $S := \{t \in [0, 1] \mid \gamma(t) \in B((0, 0), \varepsilon)\}$. Let $t_0 = \inf(S)$ and $t_1 = \sup(S)$. Define $\alpha : [0, 1] \rightarrow G \setminus \{(0, 0)\}$ as follows:

$$\alpha(t) = \begin{cases} \gamma(t) & \text{if } t \in [0, t_0] \cup [t_1, 1] \\ \frac{t_1-t}{t_1-t_0}\gamma(t_0) + \frac{t-t_0}{t_1-t_0}\gamma(t_1) & \text{if } t \in [t_0, t_1] \end{cases}$$



Clearly, $\alpha(t) \in G \setminus \{(0, 0)\}$ for all $t \in [0, t_0] \cup [t_1, 1]$. So, what remains to be shown is that the line segment connecting $\gamma(t_0)$ and $\gamma(t_1)$ is contained in $G \setminus \{(0, 0)\}$. But, note that $B((0, 0), \varepsilon) \cap X \subset G$ and it is a sector of $B((0, 0), \varepsilon)$. Further, $\gamma(t_0)$ and $\gamma(t_1)$ lie on the circular boundary of the sector. As the angle of the sector is $\pi/4$ (and not π), the chord connecting them does not pass

through the centre of the circle, namely $(0, 0)$.

□

If $(0, 0) \notin G$, then $Gr(F|_G)$ must be connected since $F|_G$ is a continuous function as $(0, 0)$ is the only point of discontinuity. We were given that G is connected, and the continuous image of a connected set is connected.

So, let us suppose that $(0, 0) \in G$. Assume $Gr(F|_G)$ is not connected. Then, there exist non-empty disjoint open sets $U, V \subset Gr(F|_G)$ s.t. $U \cup V = Gr(F|_G)$. So, without loss of generality, we can assume that $(0, 0, 0) \in U$ and $(0, 0, 0) \notin V$.

Since $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$ is open (since $\{(0, 0, 0)\}$ is closed) and we're given that U is open, their *intersection* $U \setminus \{(0, 0, 0)\}$ is also an open subset of $Gr(F|_G)$. Moreover, $U \setminus \{(0, 0, 0)\} = Gr(F|_{G \setminus \{(0, 0, 0)\}}) \cap U \setminus \{(0, 0, 0)\}$ and hence is also an open subset of $Gr(F|_{G \setminus \{(0, 0, 0)\}})$. Similarly, V is an open subset of $Gr(F|_{G \setminus \{(0, 0, 0)\}})$ as well. Then, $U \setminus \{(0, 0, 0)\} \cup V = Gr(F|_{G \setminus \{(0, 0, 0)\}})$. Therefore, $Gr(F|_{G \setminus \{(0, 0, 0)\}})$ is not connected.

However, $G \setminus \{(0, 0)\}$ is a connected set (see the in-proof lemma) and $F|_{G \setminus \{(0, 0)\}}$ is continuous, so $Gr(F|_{G \setminus \{(0, 0)\}})$ must be connected. This is a contradiction.

Therefore, regardless of whether or not $(0, 0) \in G$, the graph of the function restricted to this set is connected. □

3. We also prove that if G is a connected open subset of X , then $Gr(F|_G)$ is also path-connected.

If $(0, 0) \notin G$, the graph is path-connected. The function is continuous at all points in X except for $(0, 0)$ so it follows by the previous lemma.

Suppose $(0, 0) \in G$. Take any $r > 0$. Notice that $B_r = \{(x, y) \in X \mid x < r\}$ is an open (convex) set containing $(0, 0)$. Then $F_r = Gr(F) \cap (B_r \times [0, r])$ is an open subset of F containing $(0, 0, 0)$. Now $Gr(F|_G) = F_r \cup Gr(F|_{G \setminus \{0, 0\}})$, since $((0, 0), 0) \in F_r$. We have proven that $Gr(F|_{G \setminus \{0, 0\}})$ is path-connected, above. We will now prove that F_r is also path-connected. Since they have a non-empty intersection and are both path-connected, their union $Gr(F|_G)$ is path-connected.

Let $z_1, z_2 \in F_r$. Then $z_1 \in Gr(F)$. Thus $\exists x_1 \in X$ s.t. $z_1 = (x_1, F(x_1))$. Additionally, $z_1 \in B_r \times [0, r)$. Together, $x_1 \in B_r$ and $F(x_1) \in [0, r)$. Similarly, $\exists x_2 \in X$ s.t. $z_2 = (x_2, F(x_2))$, $x_2 \in B_r$, and $F(x_2) \in [0, r)$.

Either both points x_1 and x_2 are non-zero or exactly one is. Suppose both are non-zero. Since both points x_1 and x_2 belong to $dom(F_r)$ and $dom(F_r)$ is convex, $\gamma(t) = tx_1 + (1-t)x_2$ is a path in $dom(F_r)$, which connects x_1 and x_2 . Let $\bar{\gamma}(t) = (\gamma(t), F(\gamma(t)))$.

For $\bar{\gamma}$ to be a path, it must be continuous and the image must lie in F_r . We will first show that the image is contained in F_r . Clearly, $\bar{\gamma}(t) \in Gr(F), \forall t \in [0, 1]$. Since $\gamma(t) \in dom(F_r) = F^{-1}([0, r]) \cap B_r$, $\gamma(t) \in B_r$ and $F(\gamma(t)) \in [0, r)$. Together, $\bar{\gamma}(t) \in F_r, \forall t \in [0, 1]$. Since F is continuous on $X \setminus \{(0, 0)\}$, both γ and $F(\gamma)$ are continuous. Together, $\bar{\gamma}$ is a continuous function.

Suppose $x_1 = (0, 0)$ and $x_2 \neq (0, 0)$. Since both points belong to the convex set $dom(F_r)$, $\alpha(t) = tx_2$ is a path in $dom(F_r)$, which connects x_1 and x_2 . Let $\bar{\alpha}(t) = (\alpha(t), F(\alpha(t)))$. Just as before, for $\bar{\alpha}$ to be a path, it must be continuous and the image must lie in F_r . We will now prove the first. The function is continuous since $F|_{K_a}$ is continuous (here $a = 0$). Clearly $\bar{\alpha}(t) \in Gr(F), \forall t \in [0, 1]$. As $(a_1, a_2) \in B_r$, $a_1 < r$ so $ta_1 < r$ since $t \in [0, 1]$. Therefore, $(ta_1, 0) \in B_r, \forall t \in [0, 1]$. Now $F(ta_1, 0) = 0 \in [0, r)$. Together, $\bar{\alpha}(t) \in F_r$. \square

4. Finally, we will prove that the graph is locally path-connected. Since X is locally connected, we will show that given any open $U \subset X$, we can find an open path-connected subset of that U . Let $w = ((x, y), f(x, y)) \in U$.

Case 1: $w \neq ((0, 0), 0)$

By the product topology, we can find open $V_1 \subset X$ and $V_2 \subset \mathbb{R}$ s.t. $w \in V_1 \times V_2 \subset U$. We may assume that V_1 does not contain $(0, 0)$ (if not take $V_1 \cap X \setminus \{(0, 0)\}$). Thus, $F|_{V_1}$ is continuous and hence we can find an open set $G_1 \subset V_1$ s.t. $w \in G_1$ and $F(G_1) \subset V_2$. As X is locally connected, we can find an open connected G such that $w \in G \subset G_1$. Hence, $Gr(F|_G)$ is path-connected, by (3).

We cannot use the same principle if $w = ((0, 0), 0)$, since the function is not continuous here.

Case 2: $w = ((0, 0), 0)$.

We can find an r s.t. $F_r \subset U$. In (3) we proved that F_r is path-connected, so we have found an open path-connected set containing w that is contained in U .

Together, we have shown that $Gr(f)$ is locally path-connected. \square

3.1.2 Darboux Functions

For ensuring continuity, Darboux functions are considerably weaker than connectivity functions and the connectedness of graphs. This is illustrated by the following example.

Example 3.1.3. *There is a Darboux function $F : (0, \infty) \rightarrow (0, \infty)$ whose graph is totally disconnected.*

Proof. Let us well-order the set $(0, \infty) = \{y_\alpha : \alpha < [c]\}$. We will construct a function f point by point, $((x_\alpha, f(x_\alpha)))$ using transfinite induction over $\alpha < [c]$.

Since we have well-ordered this set, we will call the ‘first’ element y_0 . Now, since E is uncountable, we know there exists an $x_0 \in E$ such that $y_0/x_0 \notin \mathbb{Q}$ by Lemma 8. So, we define $f(x_0) = y_0$. Suppose $\beta < [c]$. Further suppose we have obtained an x_α such that $y_\alpha/x_\alpha \notin \mathbb{Q}$. Then $E \setminus \{x_\alpha : \alpha < \beta\}$ is still of size $[c]$ as we have removed atmost β elements from E . Therefore there exists an element $x_\beta \in E \setminus \{x_\alpha : \alpha < \beta\}$ such that $y_\beta/x_\beta \notin \mathbb{Q}$. Thus, by transfinite induction, we have defined x_α for all $\alpha < [c]$. Now define:

$$f(x) = \begin{cases} y_\alpha & \text{if } x = x_\alpha \text{ for some } \alpha < [c] \\ \sqrt{2} & \text{if } x \notin \{x_\alpha : \alpha < [c]\} \text{ and } x \in \mathbb{Q} \\ 1 & \text{otherwise} \end{cases}$$

Note that by our construction, $f(x)/x \notin \mathbb{Q}, \forall x \in (0, \infty)$ (this is easy enough to check for each case). Additionally, f is a surjection since every $y_\alpha \in (0, \infty)$ has at least one pre-image (namely x_α).

We will use a similar idea to construct our actual function. Let $\{(a_n, b_n) : n \in \mathbb{N}\}$ be a basis for the topology of $(0, \infty)$. Let C_1 be a Cantor set embedded in the open interval (a_1, b_1) . Let $f : C_1 \rightarrow (0, \infty)$ be a surjection such that $f(x)/x \notin \mathbb{Q}$ for every $x \in C_1$. Such an f exists since C_1 is uncountable [Lemma 8]. Now, for $n + 1 \in \mathbb{N}$, the set $(a_{n+1}, b_{n+1}) \setminus \bigcup_{k=1}^n C_k$ contains an interval (c, d) because a finite union of Cantor sets is nowhere dense (i.e. an open subset of the basis element disjoint from the union of Cantor sets exists).

Let C_{n+1} be a Cantor set embedded in (c, d) and let $f : C_{n+1} \rightarrow (0, \infty)$ be a surjection such that $f(x)/x \notin \mathbb{Q}$ for every $x \in C_{n+1}$. Similar to the previous function construction, for the remaining set $(0, \infty) \setminus \bigcup_{n \in \mathbb{N}} C_n$ define $f(x) = \sqrt{2}$ if $x \in \mathbb{Q}$ and $f(x) = 1$ if $x \notin \mathbb{Q}$.

In other words, $F : (0, \infty) \rightarrow (0, \infty)$, is defined as:-

$$F(x) = \begin{cases} f_n(x) & \text{if } x \in \bigcup_{n \in \mathbb{N}} C_n \\ \sqrt{2} & \text{if } x \notin \bigcup_{n \in \mathbb{N}} C_n \text{ and } x \in \mathbb{Q} \\ 1 & \text{otherwise} \end{cases}$$

where each function $f_n : C_n \rightarrow (0, \infty)$ is constructed using transfinite induction in an identical manner to above.

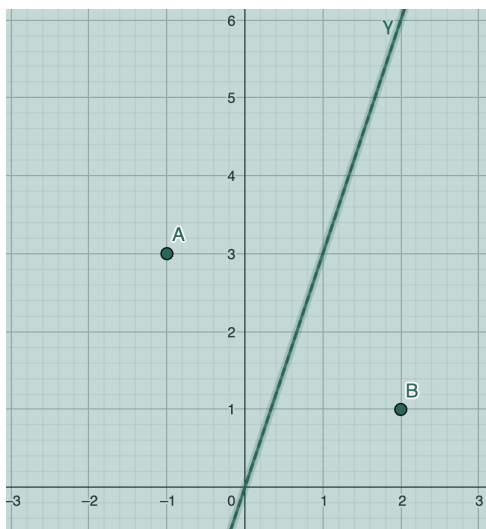
By our construction, there is at least one Cantor set embedded in any interval, so the image of any interval contains $(0, \infty)$. Since this is the codomain of the function, it is the image of any interval in \mathbb{R} . In other words, the image of a connected set (intervals are the only connected subsets of \mathbb{R}) is always connected, so the function is Darboux.

Suppose that the function was continuous. Then $F(x) \cdot 1/x = F(x)/x := G(x)$ would be a continuous function since it is the product of two continuous functions (F is defined on $(0, \infty)$). However, $F(x)/x \notin \mathbb{Q}, \forall x \in (0, \infty)$. So, the surjective function $G : (0, \infty) \rightarrow (0, \infty) \setminus \mathbb{Q}$ cannot be continuous since its domain is connected and its codomain is not (Proposition 4.11, Crossley (2005)). Since G is not continuous, F is not a continuous function.

To prove that $Gr(F)$ is totally disconnected, it remains to show that its graph does not contain

any connected sets with more than one point. Let E be an arbitrary subset of the graph containing two distinct points. For some $(a, F(a)) \in E \subset Gr(F)$, let $\alpha = F(a)/a$. If E is contained in the line $y = \alpha x$, it *cannot* be connected, since then F would be continuous on some interval. So there is some $(b, F(b)) \in E \subset Gr(F)$ with $\beta = F(b)/b, \alpha \neq \beta$.

By the density of the rationals in \mathbb{R} , there exists a rational number γ between α and β . We know that no elements of the graph lie on $y = \gamma x$, since $F(x)/x$ is always irrational. Let $A = E \cap \{(x, F(x)) \mid F(x)/x > \gamma\}$ and $B = E \cap \{(x, F(x)) \mid F(x)/x < \gamma\}$. These are disjoint open subsets of E by the subspace topology and non-empty since A and B each contain $(a, F(a))$ or $(b, F(b))$ (but not both). The union of these sets covers E since $E \subset Gr(F)$ and no elements of the graph lie on $y = \gamma x$. Thus, E is not a connected set.



So, the line $y = \gamma x$ ‘separates’ E , and the only connected subsets of the graph are singleton sets. This means that $Gr(F)$ is totally disconnected.

□

However, Darboux functions can still lend to ideas of continuity! We will modify Theorem 3.1.1, replacing the connectivity function with the weaker property of F being Darboux, but restrict the domain.

A T^3 space is one where you can ‘separate’ any point and closed set, i.e. find disjoint open sets

containing each. To read more, please refer [here](#).

Theorem 3.1.4. *Let Y be a T^3 space. Let $F : \mathbb{R} \rightarrow Y$ be a Darboux function. If F is locally connected at $(x_0, F(x_0))$, then F is continuous at x_0 .*

Proof. Let $U \subset Y$ be an open set containing $F(x_0)$. Then, $(x_0, F(x_0))$ belongs to the open set $Gr(F) \cap (\mathbb{R} \times U)$. Since F is locally connected at $(x_0, F(x_0))$, there is a connected set K such that $(x_0, F(x_0)) \in K \subset Gr(F) \cap (\mathbb{R} \times U)$. As the topology on $Gr(F)$ is inherited from the product topology on $\mathbb{R} \times Y$ and K is an open subset of $Gr(F)$, there exists an open subset A of \mathbb{R} and an open subset B of Y s.t.

- $x_0 \in A$
- $F(x_0) \in B$
- $Gr(F) \cap (A \times B) \subset K$

Now $U' = U \cap B$ is an open subset of Y (finite intersection of open sets) so $(U')^c$ is closed. Then, (as Y is T^3) there exist open sets U_1 and U_2 s.t. $F(x_0) \in U_1$, $(U')^c \subset U_2$, and $U_1 \cap U_2 = \emptyset$.

Now we can show that $\overline{U_1} \subset U'$. Given some $x \in \overline{U_1}$, let us take a sequence (x_n) completely contained in U_1 that converges to this x . Since this sequence is contained in U_1 , it is contained in the complement of U_2 (since U_1 and U_2 are disjoint). Thus, $x \in (U_2)^c$ since the set is closed (U_2 is open). If $x \in (U_2)^c$, then as $(U')^c \subset U_2$, x must be in U' . Since this x was arbitrary, this claim is true.

As A is an open subset of \mathbb{R} , $\exists r > 0 \in \mathbb{R}$ s.t. $(x_0 - r, x_0 + r) \subset A$. Now we can show that $Gr(F) \cap ((x_0 - r, x_0 + r) \times \overline{U_1}) \subset K$. Take some $(x, F(x)) \in Gr(F) \cap ((x_0 - r, x_0 + r) \times \overline{U_1})$. Then, $x \in (x_0 - r, x_0 + r) \subset A$ and $F(x) \in \overline{U_1} \subset U' \subset B$, so $(x, F(x)) \in Gr(F) \cap (A \times B) \subset K$.

Suppose that $(x, F(x)) \notin K$ for all $x \in (x_0, x_0 + r)$. Then $F((x_0, x_0 + r)) \subset Y \setminus \overline{U_1}$. Since F is a Darboux function, the set $E = F([x_0, x_0 + r])$ is connected. Further, $F([x_0, x_0 + r]) = F(\{x_0\}) \cup F((x_0, x_0 + r)) = (E \cap U_1) \cup (E \cap Y \setminus \overline{U_1})$. These are non-empty open subsets of E

by the subspace topology and disjoint (since $\overline{U_1}$ and $Y \setminus U_1$ are disjoint), whose union covers E . The existence of these sets disconnects E , which is a contradiction.

Thus, there exists $b \in (x_0, x_0 + r)$ such that $b \in \text{dom}(K)$. Since $\text{dom}(K)$ is an interval, it contains $[x_0, b]$. Using a parallel argument (suppose that $(x, F(x)) \notin K$ for all $x \in (x_0 - r, x_0)$), we can prove that there exists $a \in (x_0 - r, x_0)$ such that $[a, x_0] \subset \text{dom}(K)$.

Let $Z = (a, b)$. Then, $x_0 \in Z \subset \text{dom}(K)$, since the $\text{dom}(K)$ is a connected set. All elements of Z are in the projection of K onto \mathbb{R} , so by how we've defined $\text{dom}(K)$, for every $z \in Z$, $F(z) \in U$ (since $K \subset \text{Gr}(F)$). Thus, $F(Z) \subset U$.

Given $U \subset_{\text{open}} Y$ s.t. $F(x_0) \in U$, we found an open subset Z of \mathbb{R} ((a, b) is an open interval) s.t. $x_0 \in Z$, whose image under F was contained in U . Thus, the function is continuous at x_0 . \square

3.2 A Disconnected Complement

Until now we have been working with \mathbb{R} as a domain. To characterise continuity, we will need to further generalise, especially in the spaces we are working with.

Theorem 3.2.1. *If X is a topological space, $f : X \rightarrow \mathbb{R}$, $\text{Gr}(f)$ is connected, and $(X \times \mathbb{R}) \setminus \text{Gr}(f)$ is disconnected then f is continuous.*

Proof. Let $S = (X \times \mathbb{R}) \setminus \text{Gr}(f)$. Suppose $\text{Gr}(f)$ is connected and its complement is not connected. Then, there exist two disjoint non-empty open subsets $U, V \subset S$, such that $S = U \cup V$. By the subspace topology, $U = S \cap A$ and $V = S \cap B$, where A, B are open subsets of $X \times \mathbb{R}$.

Now, $S = U \cup V = (S \cap A) \cup (S \cap B) = (S \cap (A \cup B)) \subset A \cup B$. Additionally, since $U \subset A \setminus \text{Gr}(f)$ is non-empty and $V \subset B \setminus \text{Gr}(f)$ is non-empty, $A \setminus \text{Gr}(f)$ and $B \setminus \text{Gr}(f)$ are non-empty sets.

Further, $(A \cap B) \setminus \text{Gr}(f) = (A \setminus \text{Gr}(f)) \cap (B \setminus \text{Gr}(f)) = (S \cap A) \cap (S \cap B) = U \cap V = \emptyset$. This gives us that $A \cap B \subset \text{Gr}(f)$. Hence, since $A \cap B$ is an open subset of $X \times \mathbb{R}$, $A \cap B \subset \text{int}(\text{Gr}(f))$, which is empty by Lemma 4.

Finally, since A, B are disjoint open sets, we have that $\overline{A} \cap B = A \cap \overline{B} = \emptyset$ (by Lemma 2). Now, $\overline{A} \setminus Gr(f) = \overline{A} \cap S \subset \overline{A} \cap (A \cup B) = (\overline{A} \cap A) \cup (\overline{A} \cap B) = A \cup \emptyset = A$. Similarly, $\overline{B} \setminus Gr(f) \subset B$.

The sets $\{x\} \times (f(x), \infty)$ and $\{x\} \times (-\infty, f(x))$ are connected (they are homeomorphic to $(f(x), \infty)$ and $(-\infty, f(x))$, respectively).

For ease let $Z = \{x\} \times (f(x), \infty)$. As $Z \subset S \subset A \cup B$, we can express it as $Z = (Z \cap A) \cup (Z \cap B)$. Further, A and B are disjoint open subsets of $X \times \mathbb{R}$, and hence $Z \cap A$ and $Z \cap B$ are disjoint open subsets of Z . If both sets are non-empty, they would separate the connected set $\{x\} \times (f(x), \infty)$. Therefore, if $(x, f(x) + 1) \in A$, then $(x, f(x) + 1) \notin B$ so $\{x\} \times (f(x), \infty) \subset A$. We get the following consequences using similar arguments.

1. $(x, f(x) + 1) \in A \implies \{x\} \times (f(x), \infty) \subset A$
2. $(x, f(x) + 1) \in B \implies \{x\} \times (f(x), \infty) \subset B$
3. $(x, f(x) - 1) \in A \implies \{x\} \times (-\infty, f(x)) \subset A$
4. $(x, f(x) - 1) \in B \implies \{x\} \times (-\infty, f(x)) \subset B$

For $K, L \in \{A, B\}$, let $G_L^K := \{(x, f(x)) : (x, f(x) + 1) \in K \wedge (x, f(x) - 1) \in L\}$. Suppose $(x, f(x)) \in G_A^A \wedge G_B^A$. Then, $(x, f(x) - 1) \in A \cap B$, but $A \cap B = \emptyset$, so these sets are disjoint. Using a similar argument we can show that $G_A^A, G_B^A, G_A^B, G_B^B$ are pairwise disjoint. Clearly, $G_L^K \subset Gr(f)$. Let $(x, f(x)) \in Gr(f)$. Then, $(x, y) \in G_L^K$ since the complement of the graph is contained in $A \cup B$. Thus, $Gr(f) = G_A^A \cup G_B^A \cup G_A^B \cup G_B^B$.

We are going to show that these four sets are closed subsets of the graph. This implies that three of them are empty and one of them is the whole graph, otherwise we would be able to disconnect the graph (with two or more clopen sets). Take any net $(x_t, f(x_t))$ contained in G_L^K that converges to $(x, f(x))$. Since $(x_t, f(x_t) + 1) \in K$, we get $(x, f(x) + 1) \in \overline{K}$. With $(x, f(x) + 1) \notin Gr(f)$, $(x, f(x) + 1) \in K$ (since $\overline{K} \setminus Gr(f) \subset K$). Similarly, since $(x_t, f(x_t) - 1) \in L$, $(x, f(x) - 1) \in \overline{L} \setminus Gr(f) \subset L$. Hence $(x, f(x)) \in G_L^K$, which shows that the set G_L^K is closed.

The whole graph is equal to one of these four sets. If $Gr(f) = G_A^A$ or $Gr(f) = G_B^B$, either $B \setminus Gr(f)$ would be empty or $A \setminus Gr(f)$ would be empty. However, recalling that *both* $A \setminus Gr(f) \neq$

\emptyset and $B \setminus Gr(f) \neq \emptyset$, we conclude that either $Gr(f) = G_B^A$ or $Gr(f) = G_A^B$. Without loss of generality we may assume that $Gr(f) = G_B^A = \{(x, f(x)) : (x, f(x) + 1) \in A \wedge (x, f(x) - 1) \in B\}$.

Since A, B are open, from Lemma 7 our function is continuous if we can show that $A = \{(x, y) : f(x) < y\}$ and $B = \{(x, y) : f(x) > y\}$.

We have shown that $Gr(f) = G_B^A$, so from (1) and (4), $\{(x, y) : f(x) < y\} = \bigcup_{x \in X} \{x\} \times (f(x), \infty) \subset A$ and $\{(x, y) : f(x) > y\} = \bigcup_{x \in X} \{x\} \times (-\infty, f(x)) \subset B$.

To show that the set A is contained in $\{(x, y) : f(x) < y\}$, we will prove that it has no points that belong to any other ‘part’ of the graph. We know that A and B are disjoint, so $A \cap (\{x\} \times (-\infty, f(x))) \subset A \cap B = \emptyset$. We only need to prove now that $A \cap Gr(f) = \emptyset$. Suppose $(x, f(x)) \in A \cap Gr(f)$. Since A is open in $X \times \mathbb{R}$, by the product topology we can find open $E \subset X$ and $F \subset \mathbb{R}$ s.t. $(x, f(x)) \in E \times F \subset A$. Now since F is open in \mathbb{R} , $\exists \epsilon > 0$ s.t. $(x - \epsilon, x + \epsilon) \subset F$. Together, $(x, f(x) - \epsilon) \in A \cap B = \emptyset$, a contradiction. Therefore, $A \subset \{(x, y) : f(x) < y\}$. Similarly, we get $B \subset \{x\} \times (-\infty, f(x))$.

Now we have shown that $A = \{(x, y) : f(x) < y\}$ and $B = \{(x, y) : f(x) > y\}$, so by Lemma 7 the function f is continuous. \square

This yields the following result:

Theorem 3.2.2. *Let X be a connected space and $f : X \rightarrow \mathbb{R}$. Then f is continuous if and only if $Gr(f)$ is connected and $(X \times \mathbb{R}) \setminus Gr(f)$ is disconnected.*

Proof. If f is continuous, then $Gr(f)$ is homeomorphic to X via the domain projection map $\Theta(x, f(x)) = x$. Hence, as X is connected, $Gr(f)$ is connected. By Lemma 7, $A = \{(x, y) : f(x) < y\}$ and $B = \{(x, y) : f(x) > y\}$ are open sets. Since they are non-empty, disjoint and cover the complement of the graph, $(X \times \mathbb{R}) \setminus Gr(f)$ is disconnected.

If $Gr(f)$ is connected and $(X \times \mathbb{R}) \setminus Gr(f)$ is disconnected, by Theorem 3.2.1 f is continuous. \square

Even though we moved away from domain \mathbb{R} , previous results in this section still require

codomain \mathbb{R} . To ‘characterise continuity’ we aim to generalise as far as possible. With this in mind, see the following theorem.

(Note that this does *not* hold for \mathbb{R} , removing a point disconnects it.)

Theorem 3.2.3. *If X is a connected space, Y is a topological space such that for every $y \in Y$ the set $Y \setminus \{y\}$ is connected and has at least two elements, and $f : X \rightarrow Y$ is an arbitrary function, then $(X \times Y) \setminus Gr(f)$ is connected.*

Proof. Let $S = (X \times Y) \setminus Gr(f)$, and take two open sets $A, B \subset X \times Y$ such that $S \subset A \cup B$, $A \cap B \setminus Gr(f) = \emptyset$, and $A \setminus Gr(f) \neq \emptyset$. Then $A' := S \cap A = A \setminus Gr(f)$ and $B' := S \cap B = B \setminus Gr(f)$ are open subsets of S by the subspace topology.

The union of these sets is S : $A' \cup B' = (S \cap A) \cup (S \cap B) = S \cap (A \cup B) = S$. Additionally, we can prove that their intersection is empty:

$$\begin{aligned}
 x \in A' \cap B' &\implies x \in A' \text{ and } x \in B' \\
 &\implies x \in (A \cap S) \text{ and } x \in (B \cap S) \\
 &\implies x \in (A \cap B) \text{ and } x \in S \\
 &\implies x \in (A \cap B) \text{ and } x \in (X \times Y) \text{ and } x \notin Gr(f) \\
 &\implies x \in (A \cap B) \setminus Gr(f) \text{ and } x \in (X \times Y)
 \end{aligned}$$

This is a contradiction since we know that $A \cap B \setminus Gr(f) = \emptyset$. Thus, $A' \cap B' = \emptyset$.

Since we found two disjoint open sets A', B' whose union covers S , given A' is non-empty, showing B' is empty is sufficient for proving that the complement of the graph is connected.

Since $\{x\} \times (Y \setminus \{f(x)\})$ is homeomorphic to the connected set $Y \setminus \{f(x)\}$, it is connected as well. For ease let $Z = \{x\} \times (Y \setminus \{f(x)\})$. As $Z \subset S \subset A \cup B$, we can express it as $Z = (Z \cap A) \cup (Z \cap B)$. Further, A and B are disjoint open subsets of $X \times Y$, and hence $Z \cap A$ and $Z \cap B$ are disjoint open subsets of Z . If both sets are non-empty, they would separate $\{x\} \times (Y \setminus \{f(x)\})$, a connected set. Therefore, if $(x, y) \in A'$, then $Z \cap B = \emptyset$ so $\{x\} \times (Y \setminus \{f(x)\}) \subset A$. We get the second

consequence using a parallel argument for B' and B :

$$1. (x, y) \in A' \implies \{x\} \times (Y \setminus \{f(x)\}) \subset A$$

$$2. (x, y) \in B' \implies \{x\} \times (Y \setminus \{f(x)\}) \subset B$$

Let $E = \{x \in X : \{x\} \times (Y \setminus \{f(x)\}) \subset A\}$. We assumed that $A' \neq \emptyset$ i.e. $\exists(x, y) \in A'$, so $\{x\} \times (Y \setminus \{f(x)\}) \subset A$ by (1). By the set's definition, clearly $x \in E$. We are going to show that E is a closed set, by contradiction.

Suppose E was not a closed set. Then $\exists x' \in \overline{E} \setminus E$. Since A', B' cover S and $(x', y') \notin A'$, we know that $(x', y') \in B'$. By (2), $\{x'\} \times (Y \setminus \{f(x')\}) \subset B$. We are given that there exist distinct points $y_1, y_2 \in Y \setminus \{f(x)\}$. Since $y_1 \neq f(x)$ and $y_2 \neq f(x')$, $(x, y_1), (x, y_2) \in B$. Now, B is an open subset of $X \times Y$, so by the product topology, $\exists U_1, U_2$ open in X and $\exists V_1, V_2$ open in Y such that $(x', y_1) \in U_1 \times V_1 \subset B$ and $(x, y_2) \in U_2 \times V_2 \subset B$. The point x' is a limit point of E , so by definition $\exists a \in (U_1 \cap U_2) \cap E$. Since f is a well-defined function, without loss of generality $f(a) \neq y_1$. Now, since $a \in E$, $(a, y_1) \in A'$. However, $(a, y_1) \in U_1 \times U_2 \subset B$ and $(a, y_1) \in S$, together which gives us that $(a, y_1) \in B'$. Therefore, (a, y_1) belongs to the empty set $A' \cap B'$.

We reach a contradiction, so E is a closed set. Using a parallel argument, we can show that $F = \{x \in X : \{x\} \times (Y \setminus \{f(x)\}) \subset B\}$ is closed. Then, $X = E \cup F$, where E and F are disjoint and clopen. Since X is connected and $x \in E \neq \emptyset$, F must be empty. So, $E = X$ which means that $S = A'$ (since every $(x, y) \in A \cap Gr(f)^c = A'$ where $y \neq f(x)$). As A' and B' formed a disjoint cover of S , $B' = \emptyset$. We have thus proven that the complement of the graph is a connected set. □

□

Chapter 4

Closed Connected Graphs

Let $f : X \rightarrow Y$ be continuous, on a ‘nice’ enough domain. Let $(z_n) = (x_n, f(x_n))$ be a net contained in the graph that converges to some point (x, y) . By Lemma 1, $(x_n) \rightarrow x$. Since the function is continuous, $f(x_n) \rightarrow f(x)$ i.e. $y = f(x)$. Together, $(x_n, f(x_n)) \rightarrow (x, f(x))$, i.e. the limit point of (z_n) belongs to the graph. This actually the sequential definition of closed. So we have shown that the graph of f is closed and it is also connected. But does the other way hold? What can a closed connected graph give us for continuity?

To navigate through the multiple relationships we will see in the second section of this chapter, a table has been provided for reference:

	X	Y	Z	f_x	Quantifier	f_y	Quantifier
4.2.1.	GENERAL	LOCALLY CONNECTED	LOCALLY COMPACT	DARBOUX	$\forall x$	CTS	$\forall y$
4.2.4.	LOCALLY CONNECTED	CONNECTED & LOCALLY CONNECTED	LOCALLY COMPACT	CTS	$\forall x$	CTS	$\exists y$
4.2.5.	GENERAL	CONNECTED, LOCALLY CONNECTED, & LOCALLY COMPACT	LOCALLY COMPACT	DARBOUX	$\forall x$	CTS	$\exists y$

4.1 Relationships

Theorem 4.1.1. *Let X, Y be topological spaces and $f : X \rightarrow Y$ have a closed graph. If $E \subset Y$ and E is compact, then $f^{-1}(E)$ is closed.*

Proof. Let $x_0 \in \overline{f^{-1}(E)}$. We have a net (x_t) in $f^{-1}(E)$ which converges to x_0 . Clearly, $(f(x_t))$ is contained in the compact set E . By the compactness of E , there exists a subnet $(f(x_{\alpha(s)}))$ which converges to some $y_0 \in E$. Together, $(x_{\alpha(s)}, f(x_{\alpha(s)}) \rightarrow (x_0, y_0))$. Since $Gr(f)$ is a closed set, $f(x_0) = y_0 \in E$ so $x_0 \in f^{-1}(E)$. As this element was arbitrary we have shown that $\overline{f^{-1}(E)} \subset f^{-1}(E)$, i.e. the closure is contained in the set. Since the other direction of containment follows, $f^{-1}(E) = \overline{f^{-1}(E)}$ – the set $f^{-1}(E)$ is closed. \square

Theorem 4.1.2. *Let X be a topological space, Y be a compact space, and $f : X \rightarrow Y$ have a closed graph. Then f is continuous.*

Proof. Take any closed $E \subset Y$. Since E is a closed subset of the compact space Y , E is compact [Lemma 10]. By Theorem ??, $f^{-1}(E)$ is closed. So we have proven that the pre-image of any closed subset of Y is a closed subset of X , equivalent to f being a continuous function [by Lemma 10]. \square

Theorem 4.1.3. *Let X be a topological space, Y be a locally compact space, and $f : X \rightarrow Y$ have a closed graph. Then $W = \{x \in X : f \text{ is continuous at } x\}$ is open.*

Proof. Take any $x_0 \in W$. Since Y is locally compact, there exists an open $V \subset Y$ such that $f(x_0) \in V$ and \overline{V} is compact. Since f is continuous at x_0 (recall $x_0 \in W$), there exists an open $G \subset X$ such that $x_0 \in G$ and $f(G) \subset V$.

Notice that $f|_G : G \rightarrow V$ and $Gr(f|_G) = (G \times V) \cap Gr(f)$ is closed in $G \times V$ by the subspace topology, since $Gr(f)$ is a closed subset of $X \times Y$. By Theorem 4.1.2, $f|_G$ is continuous. From this we will prove that f is continuous on G .

Let $V' \subset Y$ be open and $g \in G$ s.t. $f(g) \in V'$. The set $V \cap V'$ is an open subset of V by the subspace topology. Since $f|_G$ is continuous, we can find an open subset of G containing x_0 , say G' , s.t. $f(G') \subset V \cap V'$. Since G is open, we can find an open $U \subset X$ s.t. $x_0 \in U \subset G$. Together, $x_0 \in U \cap G'$ and $f(U \cap G') \subset V \cap V' \subset V'$. So f is continuous on G , hence $G \subset W$.

Given an arbitrary $x_0 \in W$, we were able to find an open subset of X , G , s.t. $x_0 \in G \subset W$. Thus, W is an open subset of X . \square

Theorem 4.1.4. *Let X be a topological space, Y be a locally compact space, and $x_0 \in A \subset X$. Suppose that $f : X \rightarrow Y$ has a closed graph and $f|_A$ is continuous at x_0 . Suppose that for every open set G containing x_0 there is an open subset G' of x_0 such that for every $z \in G'$ we can find a set $E \subset G$ containing z such that $E \cap A \neq \emptyset$ and $f(E)$ is connected. Then f is continuous at x_0 .*

Proof. Let V be an open subset of Y such that $f(x_0) \in V$. Since Y is a locally compact space, there exists an open set W such that $f(x_0) \in W$ and \overline{W} is compact. Then $f(x_0) \in V \cap W =: U$ is open and $\overline{U} \subset \overline{W}$ is compact by Lemma 10 – the closed subset of a compact set is compact. Now, $\overline{U} \setminus U = \overline{U} \cap U^c$ is closed by the subspace topology (U is open) and contained in the compact set \overline{U} , so from the same it is compact. Since $Gr(f)$ is closed, Theorem ?? gives us that $f^{-1}(\overline{U} \setminus U)$ is closed. Further, $f(x_0) \in U$, so $x_0 \notin f^{-1}(\overline{U} \setminus U)$. Together, we know there exists an open set G_1 containing x_0 such that $G_1 \cap f^{-1}(\overline{U} \setminus U) = \emptyset$, and so $f(G_1) \cap (\overline{U} \setminus U) = \emptyset$. Since $f|_A$ is continuous at x_0 , there exists an open set G_2 containing x_0 such that $f(G_2 \cap A) \subset U$. Let $G = G_1 \cap G_2$. By our supposition in the theorem, there exists a neighbourhood G' of x_0 , s.t. $G' \subset G$, with the properties given: Take any $z \in G'$. Then, (1) $z \in E \subset G$, (2) $\exists a \in E \cap A$, and (3) $f(E)$ is connected. Since $E \subset G_1$ and $f(G_1) \cap (\overline{U} \setminus U) = \emptyset$, clearly $f(E) \subset U \cup (Y \setminus \overline{U})$. Finally, $E \subset G_2$, so $f(E \cap A) \subset f(G_2 \cap A) \subset U$, i.e. $f(a) \in U$.

Suppose $f(z) \notin U$. Then, $f(z) \in Y \setminus \overline{U}$. Since $a, z \in E$, we can disconnect the connected set $f(E) = (f(E) \cap U) \cup (f(E) \cap (Y \setminus \overline{U}))$, as these are two disjoint non-empty ($f(a)$ belongs to the first and $f(z)$ belongs to the second) open sets. This is a contradiction, so $f(z) \in U$. We have shown that $f(G') \subset U \subset V$, so the function is continuous at x_0 .

□

4.2 Continuity through sections

Suppose we want to further generalise the spaces we're working in. If we have information about the function restricted to certain sections, then we can characterise continuity.

We define the x -section and y -section of any function $f : X \times Y \rightarrow Z$ as $F_x : Y \rightarrow Z$ and $F_y : X \rightarrow Z$, respectively as $F_x(y) = f(x, y) = F_y(x)$.

Theorem 4.2.1. *Let X be a topological space, Y be a locally connected space, Z be a locally compact space, and $f : X \times Y \rightarrow Z$ have a closed graph. If $y_0 \in Y$,*

1. *the mapping $f_x : Y \rightarrow Z$ defined by $f_x(y) = f(x, y)$ is Darboux for every $x \in X$*
2. *the mapping $f_{y_0} : X \rightarrow Z$ defined by $f_{y_0}(x) = f(x, y_0)$ is continuous*

then f is continuous at (x_0, y_0) for all $x_0 \in X$.

Proof. We will use Theorem 4.1.4 to prove this theorem.

Let $x_0 \in X$ and $A = X \times \{y_0\}$. We define $f|_A : X \times \{y_0\} \rightarrow Z$ as $f|_A(x, y_0) = f(x, y_0)$. By (2), $f_{y_0}(x) = f(x, y_0) = f|_A(x, y_0)$ is continuous. Therefore, $f|_A$ is continuous at (x_0, y_0) . Suppose G is an open subset of $X \times Y$ containing (x_0, y_0) . By the product topology, we can find open subsets U, V of X, Y , respectively s.t. $(x_0, y_0) \in U \times V \subset G$. Since Y is a locally connected space, there exists a connected open set K such that $y_0 \in K \subset V$. Let $G' = U \times K$ and $z = (x, y) \in G'$. Let $E = \{x\} \times K$. Clearly, $z \in E$ and $y_0 \in K$, so $(x, y_0) \in E \cap A \neq \emptyset$. By (1), f_x is Darboux so since K is connected, $f_x(K) = f(E)$ is connected. Using Theorem 4.1.4, f is continuous at (x_0, y_0) . Since x_0 was arbitrary, f is continuous at (x_0, y_0) for all $x_0 \in X$. □

Corollary 4.2.1. *Let X be a topological space, Y be a locally connected space, Z be a locally compact space, and $f : X \times Y \rightarrow Z$ have a closed graph. If*

1. *the mapping $f_x : Y \rightarrow Z$ defined by $f_x(y) = f(x, y)$ is Darboux for every $x \in X$*

2. the mapping $f_y : X \rightarrow Z$ defined by $f_y(x) = f(x, y)$ is continuous for every $y \in Y$

then f is continuous.

Proof. Let $y \in Y$. By Theorem 4.2.1, the function is continuous at $f(x, y)$ for all $x \in X$. Since y was arbitrary, f is a continuous function. \square

Theorem 4.2.2. *Let Y be a locally connected space, Z is a locally compact space, and $f : Y \rightarrow Z$ be a Darboux function with a closed graph. Then f is continuous.*

Proof. Let $x_0 \in X$. We define the function $F : \{x_0\} \times Y \rightarrow Z$ as $F(x_0, y) = f(y)$. From our previous definition of the x -section: $F_{x_0} : Y \rightarrow Z$ defined as $F_{x_0}(y) = F(x_0, y) = f(y)$. It follows that, $F_{x_0} = f$, so F_{x_0} is a Darboux function. From our previous definition of the y -section: $F_y : \{x_0\} \rightarrow Z$ defined as $F_y(x_0) = F(x_0, y) = f(y)$. This function is continuous as the inverse image of any open set is $\{x_0\}$ or \emptyset ; which are both open subsets of $\{x_0\}$. If we can prove that $Gr(F)$ is closed, then Corollary 4.2.1 will give us that F is continuous. We obtain the following sets:

1. $Gr(F) = \{((x_0, y), F(x_0, y)) \mid (x_0, y) \in \{x_0\} \times Y\} = \{((x_0, y), f(y)) \mid (x_0, y) \in \{x_0\} \times Y\}$
2. $\{x_0\} \times Gr(f) = \{(x_0, (y, f(y))) \mid y \in Y\}$

Define $g : (\{x_0\} \times Y) \times Z \rightarrow \{x_0\} \times (Y \times Z)$ as $g((x_0, y), z) = (x_0, (y, z))$. We will show that g is continuous. Let $V \subset \{x_0\} \times (Y \times Z)$ be open. Let $g^{-1}(V) = U$. By the product topology definition, we can find open $Y' \subset Y$ and $Z' \subset Z$ s.t. $\{x_0\} \times (Y' \times Z') \subset V$. Then, $g^{-1}(\{x_0\} \times (Y' \times Z')) = (\{x_0\} \times Y') \times Z' \subset U$. So, we could find open $Y' \subset Y$ and $Z \subset Z'$ s.t. $(\{x_0\} \times Y') \times Z' \subset U$. By the product topology definition, U is open. Since V was arbitrary, the function is continuous. Proving g^{-1} is continuous is an almost identical argument.

Since $\{x_0\}$ is a closed subset of itself and $Gr(f)$ is given to be a closed subset of $Y \times Z$, $\{x_0\} \times Gr(f)$ is a closed subset of $\{x_0\} \times (Y \times Z)$ by the definition of product topology. Therefore $g^{-1}(\{x_0\} \times Gr(f)) = Gr(F)$ is closed by the continuity of g . By Corollary 4.2.1, F is continuous.

We will show that F_{x_0} (which is equal to f) is continuous. Let $U \subset Z$ be open. Since F is continuous, $F^{-1}(U)$ is an open subset of the domain, $\{x_0\} \times Y$. Let $(x_0, y_0) \in F^{-1}(U)$. By the product topology, there exist open $B \subset Y$ s.t. $(x_0, y_0) \in \{x_0\} \times B \subset F^{-1}(U)$. In other words, $y_0 \in B$, $F_{x_0}(y_0) = F(x_0, y_0) \in U$, B is an open subset of Y , and $F_{x_0}(B) = F(\{x_0\} \times B) \subset U$. By definition, F_{x_0} is continuous at (x_0, y_0) . Since y_0 was arbitrary, F_{x_0} is a continuous function, and f is continuous as they are equivalent. □

Theorem 4.2.3. *Let Y be a locally compact space and $f : \mathbb{R} \rightarrow Y$. If the graph is closed and connected then f is continuous.*

Proof. We proved in Lemma 3 that a function $f : \mathbb{R} \rightarrow Y$ with a connected graph must be Darboux. Additionally, \mathbb{R} is a locally connected space, Y is given to be locally compact and $Gr(f)$ is closed, so by Theorem 4.2.2, f is continuous. □

In Corollary 4.2.1, we see that a separately continuous (what we named the x and y sections) function $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ has to be continuous if its graph is closed. In previous theorems, we consider continuity given points in the codomain. However, this separate continuity can be weakened by instead only requiring at least one y -section to be continuous, as seen in the final two results of the chapter.

Further, in every theorem in section 4.2 we require Z to be locally compact. Take a look at the following example: let $Z = \mathbb{R} \times \mathbb{R}$ with the river metric [see chapter 2]. Z is *not* locally compact.

Define $f : \mathbb{R} \times \mathbb{R} \rightarrow Z$ as $f(x, y) = (x, y)$, the identity map. Let $x \in X$. Then $F_x : \mathbb{R} \rightarrow Z$ is defined as $F_x(y) = (x, y)$. We want to prove that F_x is continuous. Take $\epsilon < 0$. Then $|y_1 - y_2| < \epsilon \implies d^*((x, y_1), (x, y_2)) = |y_1 - y_2| < \epsilon$. Since x was arbitrary, all x -sections are continuous. Let $y \in Z$. Then $F_y : Z \rightarrow \mathbb{R} \times \mathbb{R}$ is defined as $F_y(x) = (x, y)$. We can show that if $y \neq 0$, then F_y is *not* continuous. From the river metric definition, $d^*((x_1, y), (x_2, y)) = |y| + |y| + |x_1 - x_2| > 2|y|$. Since we cannot make this value small given an ϵ , the function is not continuous. However, if $y = 0$, then F_0 is continuous. Take $\epsilon < 0$. Then $|x_1 - x_2| < \epsilon \implies d^*((x_1, 0), (x_2, 0)) = |0| + |0| +$

$|x_1 - x_2| = |x_1 - x_2| < \epsilon$. Therefore, every x -section of this function is continuous, and at least y -section is continuous. Further, we can show that $Gr(f)$ is closed. Let $z_n = ((x_n, y_n), f(x_n, y_n))$ be a convergent sequence in the graph i.e. $((x_n, y_n), f(x_n, y_n)) \rightarrow ((x, y), (x^*, y^*))$. Since $f(x, y) = (x, y)$ and the limit of any sequence is unique, $z_n = ((x_n, y_n), (x_n, y_n)) \rightarrow ((x, y), (x, y)) = ((x, y), f(x, y)) \in Gr(f)$. The function is continuous on the set $W = \{\mathbb{R} \setminus \{(0)\}\}$, but this is not an open subset of $\mathbb{R} \times \mathbb{R}$. Therefore, locally compact is essential for Theorem 4.1.3, and following theorems.

Theorem 4.2.4. *Let X be a locally connected space, Y be a connected and locally connected space, Z be a locally compact space, and $f : X \times Y \rightarrow Z$ have a closed graph. If*

1. *the mapping $f_x : Y \rightarrow Z$ defined by $f_x(y) = f(x, y)$ is continuous for every $x \in X$*
2. *the mapping $f_y : X \rightarrow Z$ defined by $f_y(x) = f(x, y)$ is continuous for some $y \in Y$*

then f is continuous.

Proof. Let $W := \{(x, y) \in X \times Y \mid f \text{ is continuous at } (x, y)\}$. Since Z is locally compact and $Gr(f)$ is closed, Theorem 4.1.3 gives us that W is an open subset of the domain. Let $x_0 \in X$.

- Let $D := \{y \in Y \mid f \text{ is continuous at } (x_0, y)\}$.

Since $\{x_0\} \times D = (\{x_0\} \times Y) \cap W$, by the subspace topology $\{x_0\} \times D$ is an open subset of $\{x_0\} \times Y$. Clearly, this gives us that $D \subset Y$ is open. We will show that D is also closed in Y .

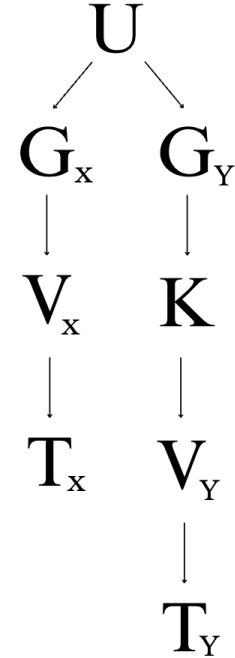
For ease of notation, we define the following sets. Let $y_0 \in D$.

- Let $\Pi := \{U \subset X \times Y \mid (x_0, y_0) \in U \text{ and } U \text{ is open}\}$.
- Let $P := \{E \subset X \times Y \mid f(E) \text{ is connected}\}$.
- Let $A := \{(x_0, y) \mid y \in Y\}$.

Let $G \in \Pi$. By the product topology we can find open $G_X \subset X$ and $G_Y \subset Y$ s.t. $(x_0, y_0) \in G_X \times G_Y \subset G$. Since Y is locally connected, we can find a connected open set K s.t. $y_0 \in K \subset G_Y$. Since $y_0 \in \overline{D}$ i.e. y_0 is a limit point of D , we can find a $y' \in K \cap D$. Since $y' \in D$, f is continuous at (x_0, y') and this point belongs to W . Since W is open we can find $V \subset W$

s.t. $(x_0, y') \in V$. The function is continuous on every point in V . Since $V \cap (G_X \times K) \subset X \times Y$ is open, we can find open $V_X \subset X$ and $V_Y \subset Y$ s.t. $(x_0, y_0) \in V_X \times V_Y \subset V \cap (G_X \times K)$. Since X and Y are locally connected, we can find connected open sets T_X s.t. $x_0 \in T_X \subset V_X$ and T_Y s.t. $y' \in T_Y \subset V_Y$.

Let $G' = T_X \times K$. Since $(x_0, y_0) \in G'$ and T_X and K are open sets of X and Y , respectively, $G' \in \Pi$ by the product topology. Let $g = (v, z) \in G'$ and $E = T_X \times T_Y \cup \{v\} \times K$. We will show that $E \in P$, i.e. $f(E)$ is connected. Notice that $T_Y \subset K$ and $v \in T_X$, so $T_X \times T_Y \cap \{v\} \times K \neq \emptyset$. Since f is well-defined, we have that $\emptyset \neq f(T_X \times T_Y \cap \{v\} \times K) \subset f(T_X \times T_Y) \cap f(\{v\} \times K)$. Now, $T_X \times T_Y$ is connected by the product topology and contained in $V_X \times V_Y \subset V$. The continuous image of a connected set is connected and f is continuous on V , so $f(T_X \times T_Y)$ is connected. From (1), f_v is continuous, so $f_v(K) = f(\{v\} \times K)$ is connected. Then $f(E)$, the union of these sets, is connected (since their intersection is non-empty) so $E \in P$. Since $z \in K$, $(v, z) \in E$. We now have $E \subset T_X \times (T_Y \cup K) \subset G_X \times G_Y \subset G$. We have $(x_0, y') \in T_X \times T_Y \subset E$ and $(x_0, y') \in A$. Together, $E \cap A \neq \emptyset$.



We define $f|_A : \{x_0\} \times Y \rightarrow Z$ as $f|_A(x_0, y) = f(x_0, y)$. By (1), $f_{x_0}(y) = f(x_0, y) = f|_A(x_0, y)$ is continuous. Therefore, $f|_A$ is continuous at (x_0, y_0) . Additionally, for an arbitrary open $G \subset X \times Y$ containing (x_0, y_0) , we found a smaller open set G' containing (x_0, y_0) such that for every $z \in G'$, $\exists E \subset G$ containing z s.t. $E \cap A \neq \emptyset$ and $f(E)$ is connected. This is the premise of Theorem 4.1.4, so f is continuous at (x_0, y_0) . By definition, $y_0 \in D$, so $\bar{D} \subset D$ i.e. D is a closed subset of Y .

By (2), we have a $y \in Y$ such that f_y is continuous. We can now apply Theorem 4.2.1 and conclude that f is continuous at (x_0, y) . So $y \in D \neq \emptyset$. Since the clopen sets D and D^c would form a separation of Y if both were non-empty then $Y \setminus D = \emptyset$. Hence, $Y = D$.

Therefore, f is continuous at (x_0, y) for all $y \in Y$. Since x_0 was arbitrary, f is continuous. □

Theorem 4.2.5. *Let X be a topological space, Y be a connected, locally connected, locally compact space, Z be a locally compact space, and $f : X \times Y \rightarrow Z$ have a closed graph. If*

1. *the mapping $f_x : Y \rightarrow Z$ defined by $f_x(y) = f(x_0, y)$ is Darboux for every $x_0 \in X$*
2. *the mapping $f_{y_1} : X \rightarrow Z$ defined by $f_{y_1}(x) = f(x, y_1)$ is continuous for some $y_1 \in Y$*

then f is continuous.

Proof. By Theorem 4.2.2, (1) can be improved to f_{x_0} is *continuous* for every $x_0 \in X$. Continuity will be required to ensure that the image of any compact set is compact.

Let $W := \{(x, y) \mid f \text{ is continuous at } (x, y)\}$. By Theorem 4.1.3, we have that W is open. Let $x_0 \in X$ and $D := \{y \in Y \mid f \text{ is continuous at } (x_0, y)\}$. Since $\{x_0\} \times D = (\{x_0\} \times Y) \cap W$, by the subspace topology $\{x_0\} \times D$ is an open subset of $\{x_0\} \times Y$. Clearly, this gives us that $D \subset Y$ is open.

Conditions match for the premise of Theorem 4.2.1 so we get that f is continuous at (x_0, y_1) i.e. $y_1 \in D$. D is open and non-empty, so if we can show that it is closed, we will have that $D = Y$ (recall that Y is connected).

Suppose D is not closed. Then let $y_0 \in \overline{D} \setminus D$. Since Y is locally compact, there is an open set U such that $y_0 \in U \subset D$ and \overline{U} is compact. As Y is locally connected, there is an open set K such that $y_0 \in K \subset U$ and K is connected. Notice that $\overline{K} \subset \overline{U}$. As a closed subset of a compact set is compact [Lemma 10], \overline{K} is compact. Thus, we have obtained a connected open set K containing y_0 , whose closure is compact. Let $E = f(\{x_0\} \times K)$. Since K is compact and f_{x_0} is continuous, $f_{x_0}(K)$ is compact. But, $f_{x_0}(K) = f(\{x_0\} \times K) = E$. Thus, E is compact. Since Z is locally compact, for each $e \in E$, $\exists U_e$ s.t. $e \in U_e$ and $\overline{U_e}$ is compact. Then $\{U_e \mid e \in E\}$ is an open cover of E . As E is compact, we can find a finite refinement of it say $\{U_{e_1}, U_{e_2}, \dots, U_{e_n}\}$. Define $M = \overline{U_{e_1}} \cup \overline{U_{e_2}} \cup \dots \cup \overline{U_{e_n}}$. Then $E \subset \text{int}(M)$. Since y_0 is a limit point of D , $D \cap K \neq \emptyset$.

Let $y' \in D \cap K$. Since $y' \in K$, $f(x_0, y') \in E \subset \text{int}(M)$, where $f(x_0, y')$ is a point of continuity. We can find an open $M' \subset X \times Y$ containing (x, y') , s.t. $f(M') \subset \text{int}(M)$. Since M is open in $X \times Y$, we can find open $G \subset X$ and $G' \subset Y$ such that $(x, y') \in G \times G' \subset M'$. In other words, we can find an open $G \subset X$ such that for every $x \in G$, $f(x, y') \in \text{int}(M)$.

Since $y_0 \notin D$, the function is not continuous at (x_0, y_0) . Suppose we can find an open $U \subset X \times Y$ s.t. $(x_0, y_0) \in U$ and $f(U) \subset M$. By Theorem 4.1.2, $f|_U : U \rightarrow M$ is continuous at (x_0, y_0) (since the graph is closed and M is compact). This is a contradiction since f is not continuous at (x_0, y_0) . Since $f(x_0, y_0) = f|_U(x_0, y_0)$, $f|_U$ is not continuous at (x_0, y_0) . Therefore, there does not exist an open set containing (x_0, y_0) whose image is completely contained in $\text{int}(M)$.

Thus, there exists a net (x_t, y_t) s.t. $(x_t, y_t) \rightarrow (x_0, y_0)$ but $f(x_t, y_t) \notin M$, and we may assume without loss of generality that for all t , $x_t \in G$, $y_t \in K$ (i.e. $\exists T$ s.t. $\forall t \geq T$, $x_t \in G$ and $y_t \in K$). Since $x_t \in G$, $f(x_t, y') \in \text{int}(M)$. Each set $f(\{x_t\} \times K)$ is connected. If $M \setminus \text{int}(M)$ was empty, then the union of non-empty sets $(Z \setminus M) \cap f(\{x_t\} \times K)$ and $\text{int}(M) \cap f(\{x_t\} \times K)$ would form a separation of the connected set $f(\{x_t\} \times K)$. Both sets are non-empty, we have explicitly found points that belong to each. Therefore, there exist y'_t such that $f(x_t, y'_t) \in M \setminus \text{int}(M)$. Since K is compact, we can find a subnet by definition. So $y'_{\alpha(s)}$ converges to some $y'' \in K$. Previously we saw that $x_t \rightarrow x_0$. Together, $(x_{\alpha(s)}, y'_{\alpha(s)}) \rightarrow (x_0, y'')$. Now, $z_s = f(x_{\alpha(s)}, y_{\alpha(s)})$ is contained in $M \setminus \text{int}(M)$ which is compact by Lemma 10.

Therefore, we can find a subnet $z_{\beta(w)}$ that converges to some $z_0 \in M \setminus \text{int}(M)$. Since $Gr(f)$ is closed, $((x_{\alpha(s)}, y'_{\alpha(s)}), (f(x_{\alpha(s)}, y'_{\alpha(s)}))) \rightarrow ((x_0, y''), z_0)$ where $z_0 = f((x_0, y''))$. Clearly $f(x_0, y'') = z_0 \in f(\{x_0\} \times K) = E \subset \text{int}(M)$. This is a contradiction, since $z_0 \notin \text{int}(M)$. Thus, $y_0 \notin \overline{D} \setminus D$. Since this y_0 was arbitrary, $\overline{D} \setminus D = \emptyset$ and $D = \overline{D}$.

Since D is closed, $Y = D$ and the function is continuous at $(x_0, y), \forall y \in Y$. Since x_0 was arbitrary, f is a continuous function. \square

Chapter 5

Special Examples

Until now we have been looking at continuity from functions with certain graph properties. For example, Theorem 4.2.4 characterises continuity for all functions from \mathbb{R} to \mathbb{R}^n ! As soon as the graph is closed, f is a continuous function.

The spaces we work with deeply influence continuity. When we move away from \mathbb{R} to even slightly more complex domains, things can go haywire. For example, there exists this discontinuous **function** constructed by Jiří Jelínek with a closed connected graph. Understanding even the construction of this function is involved and outside of the scope of this project. In this chapter we will look at two other functions with closed connected ‘nice’ graphs, that still escape continuity.

5.1 The ψ function

Example 5.1.1. *The discontinuous function $\psi : S^1 \rightarrow \mathbb{R}$, defined by $\psi(\cos(t)), \sin(t) = \frac{1}{t}$ for $t \in (0, 2\pi]$ has a closed and connected graph.*

Proof. Let us split this proof into the theorem’s three claims.

Claim 1. *The function ψ is not continuous.*

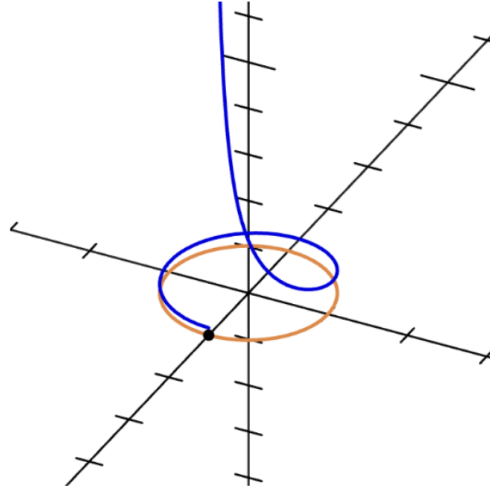


Figure 5.1: The graph of the ψ function.

Proof. Suppose that ψ is continuous. Then, for any open set V in \mathbb{R} , the pre-image of V is an open subset of S^1 . Let $V = (\frac{1}{2\pi} - \epsilon, \frac{1}{2\pi} + \epsilon)$, the open interval around $\frac{1}{2\pi}$.

We can precisely define the pre-image of the set V as follows:

$$\begin{aligned} \psi^{-1}\left(\left(\frac{1}{2\pi} - \epsilon, \frac{1}{2\pi} + \epsilon\right)\right) &= \left\{(\cos(t), \sin(t)) \mid t \in (0, 2\pi] \wedge \frac{1}{t} \in \left(\frac{1}{2\pi} - \epsilon, \frac{1}{2\pi} + \epsilon\right)\right\} \\ &= \left\{(\cos(t), \sin(t)) \mid \frac{1}{t} \in \left[\frac{1}{2\pi}, \frac{1}{2\pi} + \epsilon\right]\right\} =: U \end{aligned}$$

Note that $\psi(1, 0) = \psi((\cos(2\pi), \sin(2\pi))) = \frac{1}{2\pi}$. Therefore, $(1, 0) \in \psi^{-1}\left(\left(\frac{1}{2\pi} - \epsilon, \frac{1}{2\pi} + \epsilon\right)\right) = U$.

If U is open in S^1 , then by the subspace topology $\exists U'$ in \mathbb{R}^2 s.t.:

$$U = S^1 \cap U'.$$

Now, since U' is an open subset of \mathbb{R}^2 , $\exists \epsilon > 0$ such that $B_\epsilon((1, 0)) \subseteq U'$.

The picture above offers intuition into completing this proof. We will prove that regardless of this ϵ , there will be some points (all points where the y -coordinate is greater than 0) that belong to the intersection of the ball and S^1 , but don't actually belong to U . This will contradict the claim that $B_\epsilon((1, 0))$ is a subset of U' , and further that U is an open set.

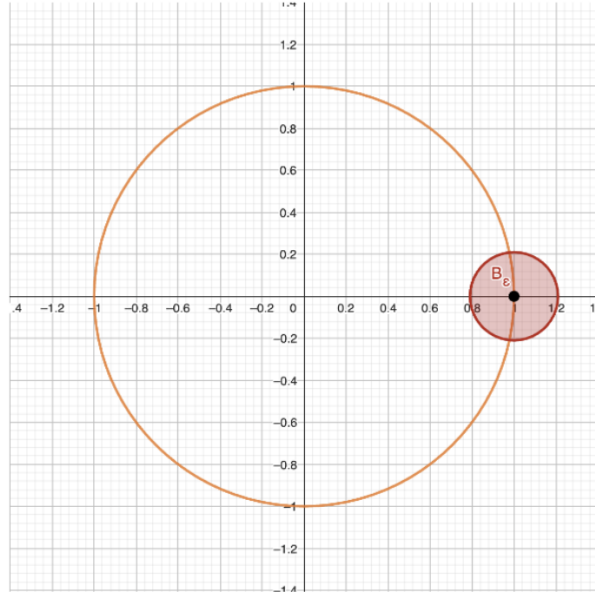


Figure 5.2: A closer look at our domain.

The Euclidean distance d_E between the point $(1, 0)$ and any point $(t, \sqrt{1-t^2})$, on the upper half of the circle is:

$$d_E\left((1, 0), (t, \sqrt{1-t^2})\right) = \sqrt{(1-t)^2 + (0 - \sqrt{1-t^2})^2} = \sqrt{1 + 1 - t^2} = \sqrt{2-t^2}.$$

Suppose $\Psi : [-1, 1] \rightarrow \mathbb{R}$ is defined as $\Psi(t) = \sqrt{2-t^2}$. This is a continuous function, so given our previous $\epsilon > 0$, we can find $\delta > 0$ s.t. $|t-1| < \delta \implies |\sqrt{2-t^2} - 0| < \epsilon$. If $t \in (1-\delta, 1]$, then $d_E\left((1, 0), (t, \sqrt{1-t^2})\right) < \epsilon$. That is, $(t, \sqrt{1-t^2}) \in B_\epsilon((1, 0), \epsilon) \subset U'$. But, $(t, \sqrt{1-t^2})$ also belongs to S^1 . Hence, if $t \in (1-\delta, 1]$, then $(t, \sqrt{1-t^2}) \in U$ (the intersection of the sets).

We defined $U = \{(\cos(t), \sin(t)) \mid \frac{1}{t} \in [\frac{1}{2\pi}, \frac{1}{2\pi} + \epsilon]\} = \{(\cos(t), \sin(t)) \mid t \in (\frac{2\pi}{1+2\pi\epsilon}, 2\pi]\}$. If $\epsilon < \frac{1}{2\pi}$, then $\frac{2\pi}{1+2\pi\epsilon} > \pi$. Therefore $\sin(t) < 0$. Thus, if $(t, \sqrt{1-t^2}) \in U$, then there exists an t such that $(\cos(t), \sin(t)) = (t, \sqrt{1-t^2})$. From this we get that $0 \leq \sqrt{1-t^2} = \sin(t) < 0$, a contradiction.

Therefore, $B_\epsilon((1, 0)) \not\subset U'$, and U is not open. The function ψ is *not* continuous. \square

Let $f : (0, 2\pi] \rightarrow \mathbb{R}^3$ be defined as $f(t) = ((\cos(t), \sin(t)), \frac{1}{t})$. Clearly $\text{Range}(f) = \text{Gr}(\psi)$. This function is continuous because each of its components is a continuous function: $y = \cos(t)$, $y = \sin(t)$, and $y = \frac{1}{t}$ for $t \in (0, 2\pi]$.

Claim 2. *The graph of ψ is closed.*

Proof. We simply need to prove that the graph of ψ contains all of its limit points to prove that it is a closed set.

Let (y_n) be a convergent sequence in $Gr(\psi)$ that converges to (a, b, c) . Since y_n belongs to $Gr(\psi)$, there exists $(x_n) \in (0, 2\pi]$ such that $(y_n) = ((\cos(x_n), \sin(x_n), \frac{1}{x_n}))$. We must prove that $(a, b, c) \in Gr(\psi)$, or in other words, there exists $t \in (0, 2\pi]$ such that $a = \cos(t)$, $b = \sin(t)$, and $c = \frac{1}{t}$.

For this proof we use Lemma 1 (where $m = 3$) – a sequence (a_n, b_n, c_n) converges to (a, b, c) iff $(a_n), (b_n), (c_n)$ converge to a, b , and c respectively.

This gives us that $(c_n) = (\frac{1}{x_n}) \rightarrow c$. This c is non-zero since the sequence is bounded below by a positive number $(\frac{1}{x_n} \geq \frac{1}{2\pi}, \forall n \in \mathbb{N})$. From the algebra of convergent sequences, we know that $(x_n) \rightarrow a \neq 0 \implies (\frac{1}{x_n}) \rightarrow \frac{1}{a}$. So, given $(z_n) = (\frac{1}{x_n}) \rightarrow c$, we have that $(\frac{1}{z_n}) = (\frac{1}{(\frac{1}{x_n})}) = (x_n) \rightarrow \frac{1}{c}$.

Then, $(\cos(x_n)) \rightarrow \cos(\frac{1}{c})$ and $(\sin(x_n)) \rightarrow \sin(\frac{1}{c})$. Together, we have that

$$(y_n) = ((\cos(x_n), \sin(x_n), \frac{1}{x_n})) \rightarrow (\cos(\frac{1}{c}), \sin(\frac{1}{c}), c)$$

Finally, we must prove that this is actually an element of the graph (i.e. $\frac{1}{c} \in (0, 2\pi]$). Since $x_n \in (0, 2\pi]$ for every $n \in \mathbb{N}$, $\frac{1}{c} \in \overline{(0, 2\pi]} = [0, 2\pi]$. However, there exists no $c \in \mathbb{R}$ such that $\frac{1}{c} = 0$. Therefore, $\frac{1}{c} \in (0, 2\pi]$, and the limit of (y_n) is an element of $Gr(\psi)$.

Since (y_n) was an arbitrary convergent sequence, the graph of ψ is a closed set. □

Claim 3. *The graph of ψ is connected.*

Proof. Suppose $Gr(\psi)$ is not connected. Then, there exist non-empty open subsets $U', V' \subset Gr(\psi)$ such that $U' \cup V' = Gr(\psi)$ and $U' \cap V' = \emptyset$. By the subspace topology, there exist open sets $U, V \subset \mathbb{R}^3$ such that $U' = Gr(\psi) \cap U$ and $V' = Gr(\psi) \cap V$ respectively.

Since f is continuous, $f^{-1}(U)$ and $f^{-1}(V)$ must be open subsets of $(0, 2\pi]$. These sets are non-empty since they are elements of the pre-image of non-empty sets U and V .

We know that U' and V' cover the range of this function. From this we will prove that the sets $f^{-1}(U)$ and $f^{-1}(V)$ cover $(0, 2\pi]$.

$$\begin{aligned}
 t \in (0, 2\pi] &\iff f(t) \in f((0, 2\pi]) = \text{Range}(f) = \text{Gr}(\psi) \\
 &\iff f(t) \in U' \cup V' \\
 &\iff f(t) \in U' \subset U \text{ or } f(t) \in V' \subset V \\
 &\iff t \in f^{-1}(U) \text{ or } t \in f^{-1}(V) \\
 &\iff t \in f^{-1}(U) \cup f^{-1}(V)
 \end{aligned}$$

We will now prove that their intersection is empty, by contradiction:

$$\begin{aligned}
 t \in f^{-1}(U) \cap f^{-1}(V) &\implies t \in f^{-1}(U) \text{ and } t \in f^{-1}(V) \\
 &\implies f(t) \in U \text{ and } f(t) \in V \\
 &\implies f(t) \in U' \text{ and } f(t) \in V' \text{ (since } f(t) \in \text{Gr}(f)) \\
 &\implies f(t) \in U' \cap V'
 \end{aligned}$$

However, $U' \cap V'$ was given to be empty. This is a contradiction, so no element can belong to $f^{-1}(U) \cap f^{-1}(V)$. We have now shown that $f^{-1}(U) \cap f^{-1}(V) = \phi$.

Therefore, we've found two disjoint open subsets of $(0, 2\pi]$ that cover it. However, the set $(0, 2\pi]$ is connected. This is a contradiction. Such sets U' and V' cannot exist, so $\text{Gr}(\psi)$ is connected. □

We have proven that, although it was a connected *and* closed graph, $\psi : S^1 \rightarrow \mathbb{R}$ is not continuous. □

5.2 Codomain ℓ^2

Example 5.2.1. *There exists a discontinuous function $f : [0, 1] \rightarrow \ell^2$ with a closed connected graph.*

Proof. Let $Y = \ell^2$. This is the set of all square-summable convergent sequences, so for any element of ℓ^2 , $\sum_{n \in \mathbb{N}} a_n^2 < \infty$.

Let (x_n) be a strictly decreasing sequence of positive numbers such that

$$\{x_n : n \in \mathbb{N}\} = \{x \in (0, 1] : \sin(\frac{\pi}{x}) = 0\}.$$

Since $\sin(n\pi) = 0$ and (x_n) is a strictly decreasing sequence, we have that $x_n = \frac{1}{n}$. Let us now define our function $f : [0, 1] \rightarrow Y$. Let $f(0) = (0, 0, \dots, 0) \in Y$ (since $\|f(0)\| = (\sum 0^2)^{\frac{1}{2}} = 0 < \infty$), the zero sequence. Take any $x \in (0, 1] = (0, x_1]$ (since $x_1 = \frac{1}{1} = 1$). There is a unique $n \in \mathbb{N}$ such that $x_{n+1} < x \leq x_n$. For each $k \in \mathbb{N}$ define:

$$f(x)(k) = \begin{cases} \sin(\frac{\pi}{x}) & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases}$$

So the image of every x is a sequence. For example, if $x = 0.7$, then $n = 1$ since $x_{1+1} = \frac{1}{2} < 0.7 \leq 1 = x_1$. So, $f(0.7) = (\sin(\frac{\pi}{0.7}), 0, 0, \dots)$. The only non-zero term in the sequence is the first term, since $n = 1$. It is clear to see that $\forall x \in (0, 1], f(x) \in Y$, as $\|f(x)\| = \sin(\frac{\pi}{x})$ (a bounded function). Thus, our function is defined.

Now it is simple to prove that $f|_{[x_{n+1}, x_n]}$ is continuous for every $n \in \mathbb{N}$. Let $y \in (x_{n+1}, x_n]$ and $(y_m) \subset (x_{n+1}, x_n]$ be a convergent sequence, s.t. $(y_m) \rightarrow y$. By how the function has been defined, $f(y_m) = (0, 0, \dots, \sin(\frac{\pi}{y_m}), 0, \dots)$ where the only non-zero term in the sequence is the n th term (which will be zero if $y_m = x_n$, for some $m \in \mathbb{N}$). Then, since both the constant zero and $\sin(x)$ functions are continuous, we have that $f(y_m) \rightarrow (0, 0, \dots, \sin(\frac{\pi}{y}), 0, \dots) = f(y)$ (again, the $\sin(\frac{\pi}{y})$ term is the n th term of the sequence). In a similar manner we can argue that the function is continuous at x_{n+1} as well (note that $\sin(\frac{\pi}{x_{n+1}}) = \sin((n+1) \cdot \pi) = 0$).

As $\bigcup_{i=1}^n [x_{i+1}, x_i] = [x_{n+1}, 1]$, $f|_{[x_{n+1}, 1]}$ is continuous for every $n \in \mathbb{N}$. Hence, $f|_{(0,1]}$ is a continuous function, i.e. f is continuous for every $x \in (0, 1]$.

Suppose $Gr(f)$ is not connected, i.e. there exist non-empty disjoint open sets $U, V \subset Gr(f)$ s.t. $U \cup V = Gr(f)$. Then, by the subspace topology there exist open subsets $U', V' \subset [0, 1] \times Y$ s.t. $U = Gr(f) \cap U'$ and $V = Gr(f) \cap V'$. We've previously shown that $f|_{(0,1]}$ is a continuous function, so $Gr(f|_{(0,1]})$ is a connected set. Then, since U and V are disjoint and cover $Gr(f)$, without loss of generality, $Gr(f|_{(0,1]}) \subset U$ (which can be proven by contradiction).

Since the complement of U is the open set V , U is a closed set. The sequence $(x_n, f(x_n)) = (x_n, (0, 0, 0, \dots)) = (x_n, f(0)) \rightarrow (0, f(0))$ as $n \rightarrow \infty$. Since U is a closed set, $(0, f(0)) \in U$. Therefore, $U = Gr(f)$, and $V = Gr(f) \setminus U = Gr(f) \setminus Gr(f) = \emptyset$, a contradiction. We have now proven that $Gr(f)$ is a connected set.

For f to be continuous at 0, for any sequence that converges to 0, the sequence of images converge to $f(0)$. This holds for (x_n) , but not necessarily any arbitrary sequence. In fact, we can prove that function is *not* continuous at 0.

Let (a_n) be a strictly decreasing sequence of positive numbers such that

$$\{a_n : n \in \mathbb{N}\} = \{x \in (0, 1] : \sin(\frac{\pi}{x}) = 1\}.$$

Since $\sin(\frac{\pi}{2} + 2n\pi) = 1$ and (a_n) is a strictly decreasing sequence, we have that $a_n = \frac{2}{1+4n}$. Take any $n \in \mathbb{N}$. Choose $k \in \mathbb{N}$ so that $x_{k+1} < a_n < x_k$. Thus, $f(a_n)(k) = \sin(\frac{\pi}{a_n}) = 1$ and so $\|f(a_n)\| = 1$ for every $n \in \mathbb{N}$.

Suppose f was continuous at 0. Then, $f(a_n) \rightarrow f(a) = f(0)$. By the definition of continuity, $f(a_n) \rightarrow f(0) \iff \|f(a_n) - f(0)\| \rightarrow 0$. Using the triangle inequality, $|\|f(a_n)\| - \|f(0)\|| \leq \|f(a_n) - f(0)\|$, so by the squeeze theorem, $|\|f(a_n)\| - \|f(0)\|| \rightarrow 0$. Again, this implies that $\|f(a_n)\| \rightarrow \|f(0)\|$. This is a contradiction, since $\|f(a_n)\|$ is the constant sequence 1, yet its limit is $\|f(0)\| = 0$. This sequence does not even converge. Therefore, the function f is not continuous at 0 so f is not a continuous function.

Let (z_n) be a convergent sequence (in $[0, 1]$). Then, to prove that the graph is closed, we must prove that if $(z_n, f(z_n)) \rightarrow (z, y)$, then $y = f(z)$. If $z \in (0, 1]$, then – since the function is continuous on this interval – $y = f(z)$. Suppose $z = 0$, i.e. $(z_n) \rightarrow 0$. We will show that $y = f(0)$. Take any $k \in \mathbb{N}$ and any $\epsilon > 0$. Since $\|f(z_n) - y\| \rightarrow 0$, we can consider continuity for each component (separately): there exists an $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$,

$$|f(z_n)(k) - y(k)| \leq \epsilon$$

Choose an $n \in \mathbb{N}$ such that $n > n_0$ and $z_n < x_{k+1}$. Then $f(z_n)(k) = 0$ (by how we've defined our function). So $|0 - y(k)| \leq \epsilon$ for every $\epsilon > 0$, equivalent to $y(k) = 0$. Since k was arbitrary, $y = (0, 0, 0, \dots) = f(0)$. With this, we have proven that the graph of f is closed. \square

Chapter 6

Conclusion

As the title indicates, we have tried to characterise continuity, using properties of the spaces our function lies in, and properties of the graph of the function. Approaching continuity by looking at the graph instead of directly proving continuity of the function arises from intuition (what was once thought to be the definition of continuity!), but also since the graph is often a more tangible idea than continuity. It is typically easier to prove a set (such as the graph) is connected or not, rather than trying to prove continuity just given the function definition. With these tools now we can simply aim to prove certain properties of the graph given a function and instantly claim it as continuous.

There were three theorems (3.1.3, 3.2.2, 4.2.3) that were in boxes in the thesis. These are the highlights of the central chapters, and generalise continuity in regards to \mathbb{R} . The fifth chapter was added to realise the power of the properties of the real line, its ‘niceness’. When we move away from it, functions struggle to achieve continuity even with well-behaved graphs.

We used the topologist’s sine curve to motivate different approaches to characterising continuity. Each chapter was created as a result of this graph and what it was missing. In fact, the graph is also not path-connected. Could replacing connected with path-connected ensure continuity in some way? Are there more such special examples that give rise to such interesting results, or at least motivate them? It would be interesting to try going down these paths in the future.

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